Topics: PDEs in other coordinate systems

In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, PDEs, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. Let $u(r,\theta)$ be a function defined on the disk of radius R. Consider the following boundary value problem for *Laplace's equation* in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad u(r,\theta+2\pi) = u(r,\theta), \qquad u(R,\theta) = 2 - 3\cos\theta + 5\sin 2\theta.$$

   (a) Assume that a solution has the form $u(r,\theta) = R(r)T(\theta)$. Plug this back in and separate variables to get an equation for R and T, including boundary conditions.

   (b) Solve the ODEs for $R(r)$ and $T(\theta)$, and determine all possible eigenvalues λ_n. Make sure to impose the additional requirement that $R(0)$ exists.

   (c) Find the general solution to Laplace's equation in polar coordinates.

   (d) Plug in $r = R$ to find the particular solution to this boundary value problem.

2. Let $u(r,\theta,t)$ be a function defined on the disk of radius $R = 1$, and for all $t \geq 0$. Consider the following initial/boundary value problem for the *heat equation* in polar coordinates:

$$u_t = c^2\,\Delta u, \qquad u(r,\theta+2\pi) = u(r,\theta), \qquad u(r,\theta,0) = 1 - r^2 + h(r,\theta)$$
$$u(1,\theta,t) = 2 - 3\cos\theta + 5\sin 2\theta.$$

   (a) Find $h(r,\theta)$, the steady-state solution.

   (b) Make the change of varibles $v(r,\theta,t) = u(r,\theta,t) - h(r,\theta)$, and re-write the PDE above, including the boundary and initial condtions, in terms of v instead of u.

   (c) Find the general solution for this *homogeneous* PDE using separation of variables. Assume that $v(r,\theta,t) = f(r,\theta)g(t)$.

   (d) Find the particular solution that satisfies the initial condition.

3. Let $u(r,\theta,t)$ be a function defined on the disk of radius $R = 1$, and for all $t \geq 0$. Consider the following initial/boundary value problem for the *wave equation* in polar coordinates:

$$u_{tt} = c^2\,\Delta u, \qquad u(r,\theta+2\pi) = u(r,\theta) \qquad u(r,\theta,0) = 1 - r^2$$
$$u(1,\theta,t) = 0, \qquad u_t(r,\theta,0) = 0.$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and both initial conditions.

   (b) Assume there is a solution of the form $u(r,\theta,t) = f(r,\theta)g(t)$. Plug this back in and separate variables to get a BVP for f and an IVP for g.

   (c) Find the general solution to this BVP for u.

   (d) Find the particular solution that additionally satisfies the initial conditions.

4. Let $u(r,\theta,\phi,t)$ be the temperature of a sphere of radius $R=\pi$. Assume that the initial temperature is constant, and that temperature does not depend on latitude or longitude. In this case, $u(r,\theta,\phi,t)=u(r,t)$, and the heat equation reduces to

$$u_t = c^2(u_{rr} + \frac{2}{r}u_r), \qquad u(\pi,t)=0, \qquad u(r,0)=T_0.$$

There is an implied boundary condition at $r=0$, that $u(0,t)$ is finite.

(a) Assume that there is a solution of the form $u(r,t)=f(r)g(t)$. Separate variables to get two equation, and ODE for g, and a (singular) Sturm-Liouville problem for f.

(b) Solve the Sturm-Liouville problem for f. [*Hint*: One could use the power series method, but a much easier way is to define $y(r)=rf(r)$, and re-write the problem in terms of y.]

(c) Find the general solution to this PDE.

(d) Find the particular solution that additionally satisfies the initial condition. Leave the formulas for coefficients in terms of integrals; no need to solve them.

5. Consider a sphere of radius $R=1$, and suppose that $u(r,\phi)$ represents a potential that depends only on the radius $r\in[0,1]$ and latitude $\phi\in[0,\pi]$. In this problem, we will solve Laplace's equation under these conditions. The boundary value problem becomes

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\phi}(u_\phi \sin\phi)_\phi = 0, \qquad u(1,\phi)=f(\phi).$$

(a) Assume $u(r,\phi)=R(r)Y(\phi)$. Plug this back in and separate variables by multiplying both sides by r^2/RY.

(b) Change variables by letting $x=\cos\phi$ for $-1<x<1$ and $y(x)=Y(\arccos(x))=Y(\phi)$. Then derive the following equations:

$$-((1-x^2)y')' = \lambda y, \qquad (r^2R')' = \lambda R,$$

(c) The equation for $y(x)$ should look familiar – it is *Legendre's equation* (see HW 2, 4, and 8). Recall that it has bounded, continuous solutions on $[-1,1]$ when

$$\lambda_n = n(n+1), \qquad y_n(x) = P_n(x), \quad n=0,1,2,\dots.$$

These are the eigenvalues and eigenfunctions of the related (singular) Sturm-Liouville problem (see HW 8). Carry out the details of deriving the general solution to Laplace's equation, which will be

$$u(r,\phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi), \qquad \text{where} \qquad c_n = \frac{1}{||P_n||^2}\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\, d\phi.$$

(d) If $f(\phi)=\sin\phi$, find an approximate, *fourth-order* solution. That is, truncate it after the $n=4$ term. The Legendre polynomials can be derived from the formula

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left[(x^2-1)^n\right],$$

but feel free to look them up online for this part.