Lecture 6.4: Solving PDEs with Fourier transforms

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The Fourier transform of a single variable function

Definition

Recall that the Fourier transform of a function $f(x)$ is defined by

$$\mathcal{F}(f) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Let $\mathcal{S}$ be subspace of $\mathcal{C}^{\infty}(\mathbb{R})$ consisting of functions that decay as $x \to \pm\infty$ faster than any polynomial:

$$\mathcal{S} = \left\{ f \in \mathcal{C}^{\infty}(\mathbb{R}) : \left| \frac{d^k f}{dx^k} \right| \leq C|x|^{-n} \text{ as } |x| \to \infty, \qquad \forall k \in \mathbb{N},\ n \in \mathbb{Z} \right\}.$$

This is the Schwartz class of functions, and $f \in \mathcal{S}$ iff $\widehat{f} \in \mathcal{S}$.

In other words, the Fourier transform is a linear operator on the space of Schwartz functions.

Convolution theorem

For functions f and g,

$$\mathcal{F}(f * g)(\omega) = \widehat{f}(\omega)\widehat{g}(\omega).$$

By taking the inverse Fourier transform of both sides, it follows that

$$(f * g)(x) = \mathcal{F}^{-1}\big(\widehat{f}(\omega)\widehat{g}(\omega)\big).$$

The Fourier transform of a multivariate function

Definition

For a function $u(x,t)$ of two variables, define its Fourier transform by

$$\mathcal{F}(u) = \widehat{u}(\omega, t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x}\, dx.$$

Remark

The Fourier transform turns x-derivatives into multiplication, and leaves t-derivatives unchanged:

- $(\mathcal{F}u_x)(\omega, t) = (i\omega)\, \widehat{u}(\omega, t)$
- $(\mathcal{F}u_{xx})(\omega, t) = (i\omega)^2\, \widehat{u}(\omega, t)$
- $(\mathcal{F}u_t)(\omega, t) = \widehat{u}_t(\omega, t)$.

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Fourier transform of the Gaussian function

Example 1

Compute the Fourier transform of a Gaussian function $u(x) = e^{-ax^2}$, where $a > 0$.

Solving an ODE with the Fourier transform

Example 2

Solve the following ODE for $u(x)$, given some forcing term $f \in \mathcal{S}$:

$$u'' = u + f(x).$$

Solving a PDE with the Fourier transform

Example 3

Solve the following Cauchy problem for the heat equation, given some $f(x) \in \mathcal{S}$:

$$u_t = c^2 u_{xx}, \qquad u(x,0) = f(x).$$