Topics: Spanning, linear independence, basis

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K.

1. Show that $B = \{x_1, \dots, x_n\}$ is a basis if and only if every $x \in X$ can be written *uniquely* using the vectors in B.

2. Let S be a linearly independent subset of X. Prove the *exchange property*: For any nonzero $x_0 \in \text{Span}(S)$, there exists $x_1 \in S$ such that $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\text{Span}(S)$.

3. Let $S = \{s_1, \dots, s_k\} \subset X$. Prove that the following are equivalent:

   (a) S is a *minimal* spanning set.

   (b) S is a *maximal* linear independent set.

   (c) S is a *basis*.