Topics: Quotients

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K.

1. If Y is a subspace of X, then two vectors $x_1, x_2 \in X$ are *congruent modulo* Y, denoted $x_1 \equiv x_2 \bmod Y$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let X/Y denote the set of equivalence classes. We can make X/Y into a vector space by defining addition and scalar multiplication as follows:

$$\{x\} + \{z\} := \{x + z\}, \qquad a\{x\} := \{ax\}.$$

   Show that these operations are *well-defined*. That is, they do not depend on the choice of congruence class representatives.

2. Let Y be a subspace of X. Prove that X is isomorphic to $Y \times X/Y$ by defining an explicit map and showing that it is linear and a bijection.