

Topics: The isomorphism theorems

Do: Answer the following questions.

1. Let $f: X \rightarrow U$ be a linear map between vector spaces. In this problem, you will prove the *first isomorphism theorem*, which says that $X/\text{Ker}(f) \cong \text{Im}(f)$, where $\text{Im}(f) = f(X)$.

(a) Let $Z = \text{Ker}(f)$, and $x+Z = \{x+z \mid z \in Z\} = \bar{x}$ be the equivalence class containing x ; either notation is fine, feel free to use either. Define a linear map

$$\iota: X/Z \longrightarrow \text{Im}(f), \quad x+Z \longmapsto f(x).$$

Carefully give a formal definition of what it means for ι to be well-defined, and then prove that it is.

(b) Prove that ι is linear, one-to-one, and onto, and conclude that $X/\text{Ker}(f) \cong \text{Im}(f)$.

2. In this problem, you will prove the main part of the second isomorphism theorem, sometimes called the *correspondence theorem*. Let Z be a subspace of X .

(a) Show that if $Z \leq Y \leq X$, then Y/Z is a subspace of X/Z .

(b) Show that *every* subspace of X/Z arises in this manner.

In this problem, it is easier to use “coset notation” $x+Z$ instead of \bar{x} , because you may need to speak of both equivalence classes $x+Y$ and $x+Z$.

3. Prove the *third isomorphism theorem*: if $Z \leq Y \leq X$, then $(X/Z)/(Y/Z) \cong X/Y$. [*Hint*: Define a map $X/Z \rightarrow X/Y$, show it is linear, onto, and has kernel Y/Z . Then apply the first isomorphism theorem.]