

Topics: The dual of a vector space

Do: Answer the following questions.

1. Let X be a vector space over a field K and let X' be the set of linear functions from X to K , also known as the *dual space* of X .
 - (a) Let $x_1, \dots, x_n$ be a basis for X . Define $\ell_j \in X'$ by $\ell_j(x_i) = \delta_{ij}$. Show that $\ell_1, \dots, \ell_n$ is a basis for X' ; it is called the *dual basis* of $x_1, \dots, x_n$.
 - (b) Find the dual basis of $x_1 = (1, -1, 3)$, $x_2 = (0, 1, -1)$, and $x_3 = (0, 3, -2)$ in $X = \mathbb{R}^3$.
 - (c) Express the scalar function $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, ℓ_1, ℓ_2, ℓ_3 , from Part (b).
2. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ over $\mathbb{R}$, with degree ≤ 2 . Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j: \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad \ell_j(p) = p(\xi_j) \quad \text{for } j = 1, 2, 3.$$

- (a) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}'_2$.
- (b) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}'_2$.