Topics: The transpose of a linear map

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K.

1. Let $T\colon X \to U$ be a linear map.
   (a) Show that the *image* $T(X)$ (i.e., the *range*, R_T) is a subspace of U.
   (b) Show that the *preimage* of a subspace $V \leq U$, denoted
   $$T^{-1}(V) := \big\{x \in X \mid Tx \in V\big\},$$
   is a subspace of X.
   (c) Give a direct proof (i.e., without using the isomorphism theorems) that T is one-to-one if and only if it has trivial nullspace.

2. Recall that the *transpose* of a linear map $T\colon X \to U$ is the linear map $T\colon U' \to X'$ between the dual spaces such that $(\ell, Tx) = (T'\ell, x)$ for all $x \in X$ and $\ell \in U'$. In class, we showed that $(ST)' = T'S'$. Show the following.
   (a) $(S+T)' = S' + T'$, for $S, T\colon X \to U$.
   (b) $I' = I$, where $I\colon X \to X$ is the identity map.
   (c) $(T^{-1})' = (T')^{-1}$.

3. Let $T\colon X \to U$ be a linear map with range R_T and nullspace N_T. Recall that the *annihilator* of a subspace $Y \leq X$ is
   $$Y^{\perp} = \big\{\ell \in X'\colon (\ell, y) = 0,\ \forall y \in Y\big\}.$$
   In class, we showed that $R_T^{\perp} = N_{T'}$. As an immediate corollary, applying this statement to $T'\colon U' \to X'$ gives the analogous result $R_{T'}^{\perp} = N_T$. Give a alternate direct proof of this fact, akin to what we did in class for $R_T^{\perp} = N_{T'}$.