Topic: Multilinearity

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K. Recall that a k-linear form $f\colon X^k \to K$ is:

- *symmetric* if $f(x_1,\dots,x_k)=\pi\cdot f(x_1,\dots,x_k):=f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(k)})$ for all $\pi\in S_k$,
- *skew-symmetric* if $\tau\cdot f(x_1,\dots,x_k)=-f(x_1,\dots,x_k)$ for all transpositions $\tau=(ij)\in S_k$,
- *alternating* if $f(x_1,\dots,x_k)=0$ whenever $x_i=x_j$.

1. Let f be a bilinear form over a vector space X with basis $\{x_1,x_2\}$.
   (a) Assume f is alternating. Determine a formula for $f(u,v)$ in terms of each $f(x_i,x_j)$ and the coefficients used to express u and v with this basis.
   (b) Repeat Part (a) but assume that f is symmetric, and that $f(x,x)=0$ for all $x\in X$.
   (c) Repeat Part (a) but now assume that X is 3-dimensional, with basis $\{x_1,x_2,x_3\}$.

2. Let $A=(c_1,\dots,c_n)$ be an $n\times n$ matrix (c_i is a column vector), and let B be the matrix obtained from A by adding k times the i^{th} column of A to the j^{th} column of A, for some $i\neq j$. Show that $\det A=\det B$. You may assume that the determinant is an alternating n-linear form.

3. Let f be an alternating k-linear form. Show that if $y_1,\dots,y_k$ are linearly dependent, then $f(y_1,\dots,y_k)=0$. Then give an explicit counterexample to show how the converse fails.