Topic: Determinant and trace

Do: Answer the following questions.

1. The following matrix is called a (4×4) *Hadamard matrix*:

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Note that the "box" formed by the four row (or column) vectors is a hypercube in $\mathbb{R}^4$. Using this information alone—purely a geometric argument—find $|\det \boldsymbol{H}|$.

2. Using linearity of each row, the determinant of an $n \times n$ matrix can be written as a sum of determinants of no more than $n!$ matrices that have *exactly one non-zero entry in each row and column.* For example, a 3×3 determinant breaks up as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \cdots.$$

From here, it is simple to compute each individual determinant. Compute the determinant of each of the following matrices using this method. Only include the non-zero terms.

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Use your answer to the first part to derive a "shortcut formula" for the determinant of any 3×3 matrix. [*Hint*: Write out the augmented 3×6 matrix $[\boldsymbol{A}|\boldsymbol{A}]$ and draw some "diagonal lines."]

3. Recall that the *trace* of an $n \times n$ matrix is $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$.

   (a) Let A be an $m \times n$ matrix, and B be an $n \times m$ matrix. Show that the $m \times m$ matrix AB and the $n \times n$ matrix BA have the same trace.

   (b) If A is an $n \times n$ matrix, derive a formula for $\operatorname{tr}(A^T A) = \operatorname{tr}(AA^T)$ in terms of a_{ij}.