Topic: Eigenvalues.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

1. Find the eigenvalues and eigenvectors for the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

2. The *characteristic polynomial* of $\boldsymbol{A}$ is $\chi_{\boldsymbol{A}}(t) = \det(t\boldsymbol{I} - \boldsymbol{A})$. Suppose this factors as

$$\chi_{\boldsymbol{A}}(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

   (a) Plug in $t = 0$ and find a formula for $\det \boldsymbol{A}$ in terms of the eigenvalues of $\boldsymbol{A}$.

   (b) The *trace* of $\boldsymbol{A}$, denoted $\operatorname{tr} \boldsymbol{A}$, is the sum of the diagonal entries, which is also equal to the sum of the eigenvalues. If $\boldsymbol{A}$ is 2×2, then

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(tI - \boldsymbol{A}) = t^2 - (a + d)t + (ad - bc) = 0.$$

   Write a formula for the characteristic polynomial of a 2×2 matrix in terms of $\det \boldsymbol{A}$ and $\operatorname{tr} \boldsymbol{A}$.

   (c) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix with characteristic polynomial $\chi_{\boldsymbol{A}}(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$. Describe $\det \boldsymbol{A}$ and $\operatorname{tr} \boldsymbol{A}$ in terms of the c_i's.

   (d) Explain why $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$ is impossible for $n \times n$ matrices.

3. Suppose $\boldsymbol{A}$ is a 3×3 matrix with eigenvalues 0, 3, and 5, with respective eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

   (a) Give a basis for the nullspace and a basis for the column space.

   (b) Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{v} + \boldsymbol{w}$. Then, find all solutions.

   (c) Explain why $\boldsymbol{Ax} = \boldsymbol{u}$ has no solution.