Topic: Eigenvectors and generalized eigenvectors.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

The *characteristic polynomial* of an $n \times n$ matrix A is $p_A(t) := \det(tI - A)$. By the *Cayley Hamilton theorem*, $p_A(A) = 0$. The *minimal polynomial* is the smallest-degree monic polynomial $m_A(t)$ for which $m_A(A) = 0$, and it must divide $p_A(t)$.

1. Consider the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

   (a) Find the characteristic and minimal polynomials of each.

   (b) Diagonalize each matrix into $\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T$, where $\boldsymbol{Q}$ is a (real-valued) *orthogonal* matrix.

   (c) Find *all* orthogonal matrices that diagonalize $\boldsymbol{A}$. How many will diagonalize $\boldsymbol{B}$?

2. Do the following for the matrix $\boldsymbol{A}$ from the previous worksheet, and then repeat it for $\boldsymbol{B}$.

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

   (a) For each eigenvalue λ, compute $\dim N_{(\boldsymbol{A}-\lambda\boldsymbol{I})^j}$ for $j = 1, 2, 3, \dots$.

   (b) Find the characteristic and minimal polynomials, and all (genuine) eigenvectors.

   (c) Find a basis $\mathcal{B}$ of $\mathbb{C}^4$ consisting of generalized eigenvectors, so that the matrix with respect to this basis is $\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$, where J is a *Jordan matrix*. This means that $\boldsymbol{J}$ is block-diagonal formed from *Jordan blocks* $\boldsymbol{J}_\lambda$; see above.

3. If $A \colon X \to X$ is a linear map, then a subspace $Y \subseteq X$ is A-*invariant* if $A(Y) \subseteq Y$. Show that for any scalar $\lambda \in K$, not necessarily an eigenvalue, the subspace Y is A-invariant if and only if it is $(A - \lambda I)$-invariant.