Topic: Jordan canonical form.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

1. Consider the following Jordan blocks with eigenvalue λ:

$$\boldsymbol{J}_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_3 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

   For each $\boldsymbol{J}_i$, compute $\boldsymbol{J}_i^2$ and $\boldsymbol{J}_i^3$. Guess the form of $\boldsymbol{J}_i^k$. Set $k = 0$ to find $\boldsymbol{J}_i^0$ and $k = -1$ to find $\boldsymbol{J}_i^{-1}$.

2. Let $\boldsymbol{A}$ be a 7×7 matrix over $\mathbb{C}$ with minimal polynomial $m(t) = (t-1)^3(t-2)^2$.

   (a) List all possible Jordan canonical forms of A up to similarity.

   (b) For each matrix from Part (a), find the rank of $(\boldsymbol{A} - \boldsymbol{I})^k$ and $(\boldsymbol{A} - 2\boldsymbol{I})^k$, for $k \in \mathbb{N}$.

3. Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix $\boldsymbol{A}$ is *nilpotent* if $\boldsymbol{A}^k = 0$ for some $k \in \mathbb{N}$, and $\boldsymbol{A}$ is *idempotent* if $\boldsymbol{A}^2 = \boldsymbol{A}$.

   (a) Prove that if $\boldsymbol{A}^k = \boldsymbol{A}$ for some integer $k > 1$, then $\boldsymbol{A}$ is diagonalizable.

   (b) Prove that idempotent matrices are similar if and only if they have the same trace.

   (c) Prove that if $\boldsymbol{A}$ is nilpotent, then $A^n = 0$.

   (d) Prove that if $\boldsymbol{A}$ is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine A up to similarity.

   (e) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_{\boldsymbol{A}} = \dim N_{\boldsymbol{B}}$. Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar. Show by example that this can fail for 7×7 matrices.