Topic: Adjoints.

Do: Answer the following questions.

1. Let $A\colon X \to U$ be a linear map between finite-dimensional inner product spaces. Recall that its *adjoint* is a linear map $A^*\colon U \to X$ such that $(Ax, u) = (x, A^*u)$ for all $x \in X$ and $u \in U$.

   (a) Give a direct proof that $R_A^\perp = N_{A^*}$, by showing that these are equal as sets.

   (b) Establish the following equalities as simple corollaries:

   (i) $R_{A^*} = N_A^\perp$ (ii) $N_A = R_{A^*}^\perp$ (iii) $R_A = N_{A^*}^\perp$.

   (c) Show that A maps R_{A^*} bijectively onto R_A.

2. Let x_1, x_2 be a basis of $X = \mathbb{R}^2$, and $\ell_1, \ell_2 \in X'$ the dual basis. Carry out the steps below for the linear map $A\colon X \to X$ defined by $A(x_1) = x_1$ and $A(x_2) = x_1 + x_2$ with respect to the standard dot product, and then with respect to each of the following inner products:

$$\langle x, y\rangle := \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{and} \qquad \langle x, y\rangle := \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   (a) Find $v_i \in X$ for which $\ell_i = \langle -, v_i\rangle$, for $i = 1, 2$.

   (b) Find $y_i \in X$ for which $A'(\ell_i) = \ell_i \circ A = \langle -, y_i\rangle$, for $i = 1, 2$.

   (c) Find the adjoint $A^*\colon X \to X$ with respect to this inner product.