Topic: Singular value decomposition, pseudo-inverses.

Read: *Spectral theorems and singular value decomposition (SVD)*, by Shuhong Gao.

Do: Answer the following question.

1. Consider the following matrices:

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}, \qquad A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}, \qquad AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}.$$

   (a) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors v_1, v_2 of A^TA.

   (b) For the $\sigma_i \neq 0$, compute $u_i = Av_i/\sigma_i$ and verify that indeed $||u_i|| = 1$. Find the other u_i by computing the other unit eigenvector of AA^T.

   (c) Construct the singular value decomposition (SVD), $A = U\Sigma V^T$.

   (d) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.

   (e) Describe *all* matrices that have the same four fundamental subspaces.

   (f) Find a left, right, and pseudoinverse of A, or explain why it doesn't exist.

2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

   (a) Construct the singular value decomposition of A.

   (b) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.

   (c) Find a left inverse, right inverse, and pseudoinverse of A, or explain why it doesn't exist.