Topic: Singular value decomposition, pseudo-inverses.

Read: *Spectral theorems and singular value decomposition (SVD)*, by Shuhong Gao.

Do: Answer the following question.

1. Let $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map defined by $f(x) = Mx$ for $x \in \mathbb{R}^4$ where $M = ABC$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

   (a) Define the adjoint map $f^*\colon \mathbb{R}^3 \to \mathbb{R}^4$ (under the standard Euclidean inner product) and express it in terms of M.

   (b) Find a singular value decomposition (SVD) of M. (*Hint*: Observe that A^TA and C^TC are diagonal.)

   (c) Find all $x \in \mathbb{R}^4$ with $\|x\| = 1$ so that $\|Mx\|$ is maximized.

   (d) Describe the eigenvalues and eigenvectors of M^TM.

   (e) Find the least square solution for $Mx = b$ with $\|x\|_2$ minimal where $b = (1, 1, 1)^T$. (*Hint*: Use the pseudo-inverse of M.)