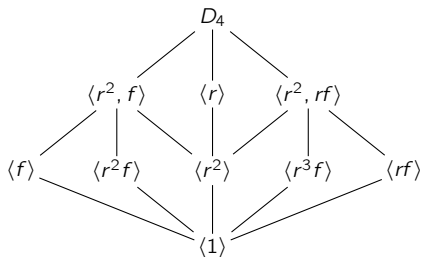
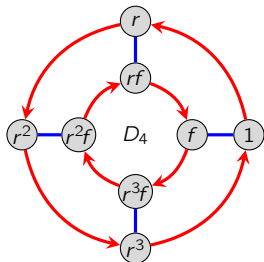
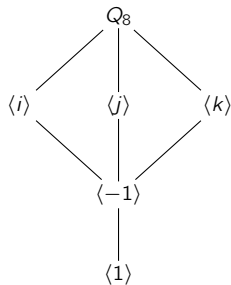
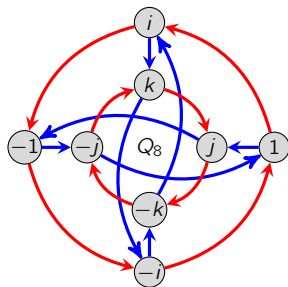


Two nonabelian groups of order 8



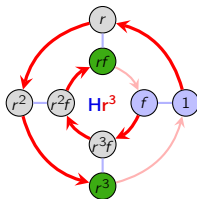
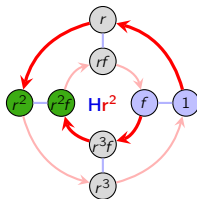
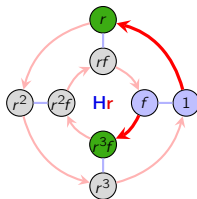
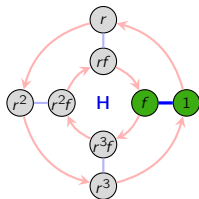
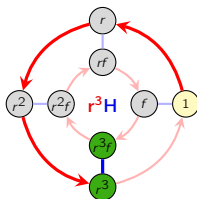
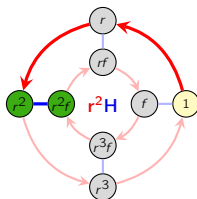
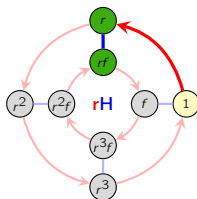
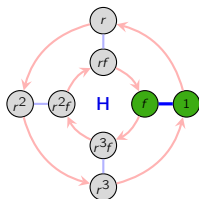
Left vs. right cosets

Definition

Let $H \leq G$. Given $x \in G$, its **left coset** xH and **right coset** Hx are:

$$xH = \{xh \mid h \in H\},$$

$$Hx = \{hx \mid h \in H\}.$$



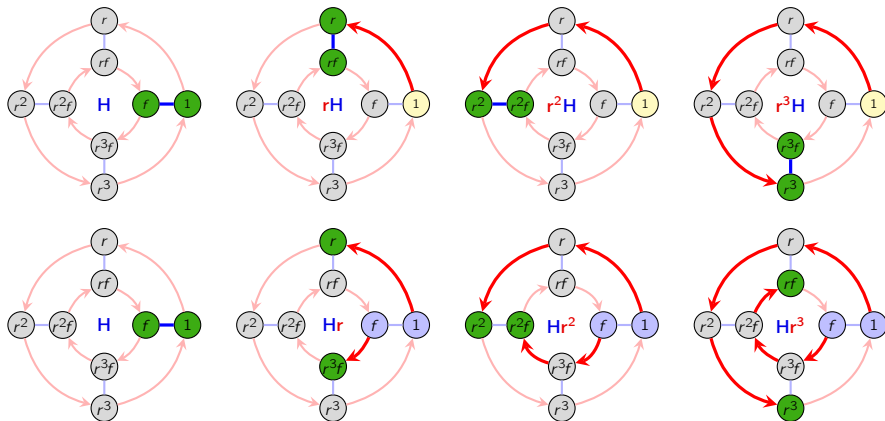
Left vs. right cosets

H r^2H rH r^3H

f	r^2f	rf	r^3
1	r^2	r	r^3f

H Hr^2

f	fr^2	fr^3	r^3	Hr^3
1	r^2	r	fr	Hr

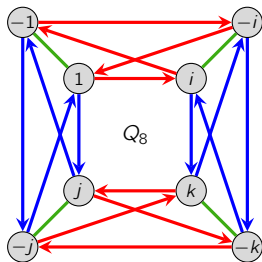


Quotients

Key idea

The quotient of G by a subgroup H exists when the (left) cosets of H form a group.

Here is the quotient of $G = Q_8$ by the subgroup $H = \langle -1 \rangle = \{1, -1\}$.



	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

$$Q_8 / \langle -1 \rangle \cong V_4$$

	± 1	$\pm i$	$\pm j$	$\pm k$
± 1	± 1	$\pm i$	$\pm j$	$\pm k$
$\pm i$	$\pm i$	± 1	$\pm k$	$\pm j$
$\pm j$	$\pm j$	$\pm k$	± 1	$\pm i$
$\pm k$	$\pm k$	$\pm j$	$\pm i$	± 1

Quotients

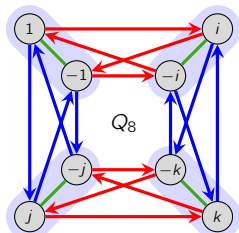
Denote the set of left cosets of H in G by

$$G/H := \{xH \mid x \in G\}.$$

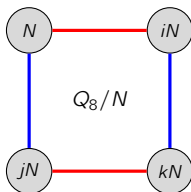
Key idea

The **quotient** of G by a subgroup H exists when the (left) cosets of H form a group.

This is well-defined precisely when **H is normal**. (Left and right cosets coincide.)



Cluster the
left cosets of N

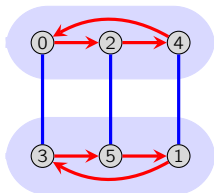


Collapse cosets
into single nodes

	N	iN	jN	kN
N	N	iN	jN	kN
iN	iN	N	kN	jN
jN	jN	kN	N	iN
kN	kN	jN	iN	N

Elements of the quotient
are cosets of N

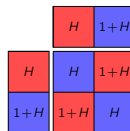
Quotients



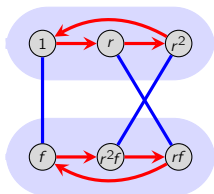
Cluster the
left cosets of $H \leq \mathbb{Z}_6$



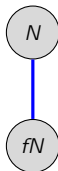
Collapse cosets
into single nodes



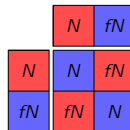
Elements of the quotient
are cosets of H



Cluster the
left cosets of $N \leq D_3$



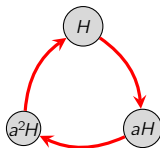
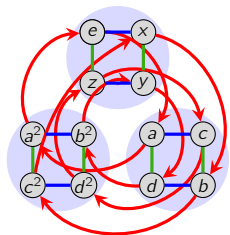
Collapse cosets
into single nodes



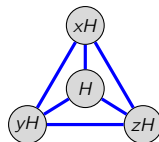
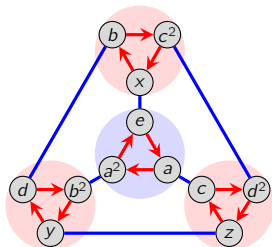
Elements of the quotient
are cosets of N

Quotients

Let's revisit $N = \langle (12)(34), (13)(24) \rangle$ and $H = \langle (123) \rangle$ of A_4 .

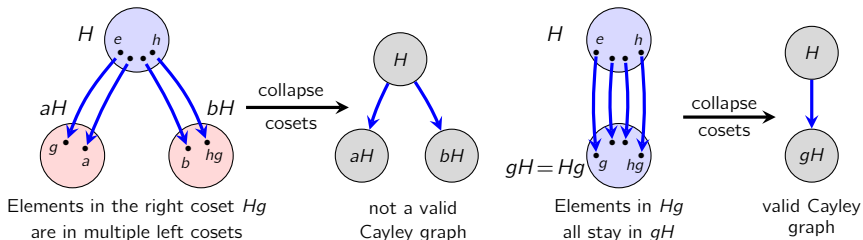


	H	aH	a^2H
H	H	aH	a^2H
aH	aH	a^2H	H
a^2H	a^2H	H	aH



When and why the quotient process works

In the following: *the right coset Hg are the nodes at the "arrowtips".*



Key idea

If H is **normal subgroup** of G , then the quotient group G/H exists.

If H is not normal, then following the blue arrows from H is **ambiguous**.

In other words, it **depends on our where we start within H** .

What does it mean to “multiply” two cosets?

Quotient theorem

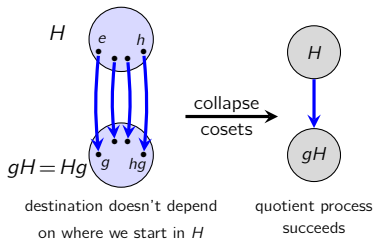
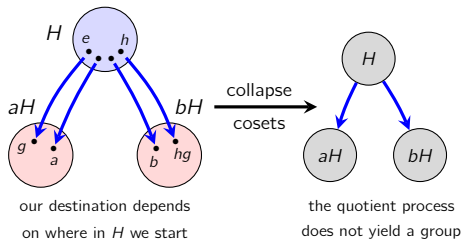
If $H \trianglelefteq G$, the set of cosets G/H forms a group, with binary operation

$$aH \cdot bH := abH.$$

It is clear that G/H is closed under this operation.

We have to show that this operation is **well-defined**.

By that, we mean that it *does not depend on our choice of coset representative*.



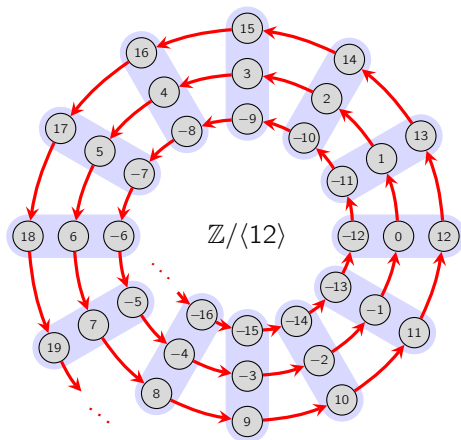
A familiar example

Consider the subgroup $H = \langle 12 \rangle = 12\mathbb{Z}$ of $G = \mathbb{Z}$.

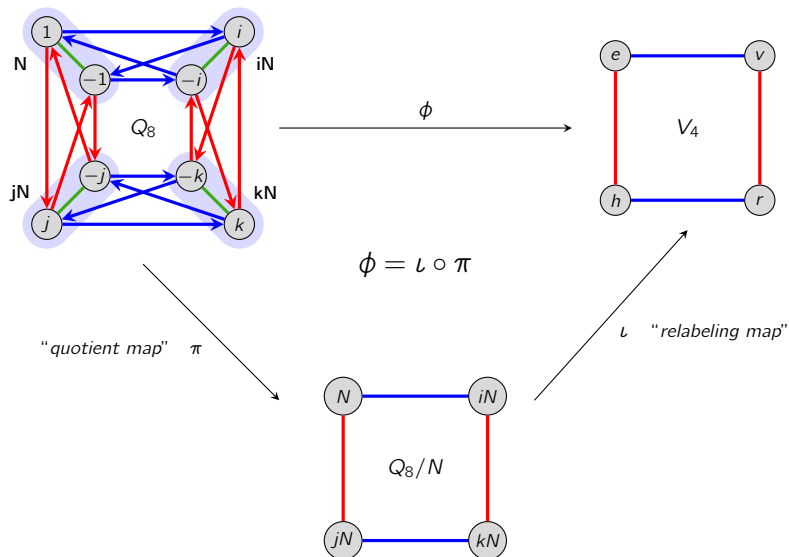
The cosets of H are the **congruence classes** modulo 12.

Since this group is additive, the condition $aH \cdot bH$ becomes $(a + H) + (b + H) = a + b + H$:

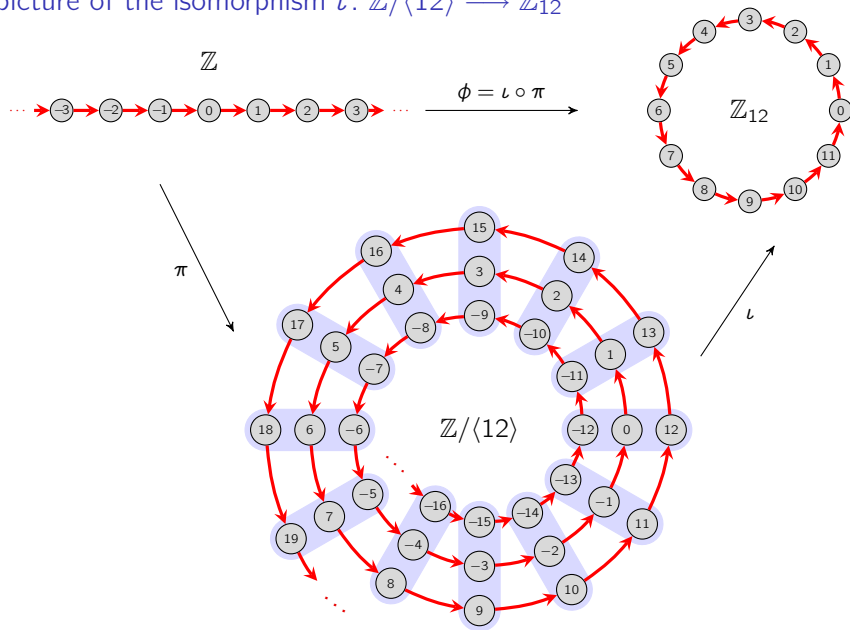
"(the coset containing a) + (the coset containing b) = the coset containing $a + b$."



Visualizing the FHT via Cayley graphs

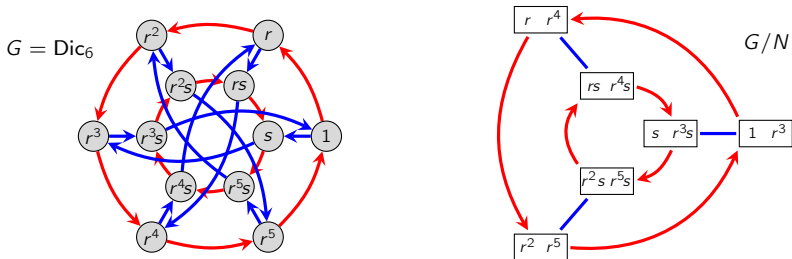


A picture of the isomorphism $\iota: \mathbb{Z}/\langle 12 \rangle \longrightarrow \mathbb{Z}_{12}$



The correspondence theorem: subgroups of quotients

Compare $G = \text{Dic}_6$ with the quotient by $N = \langle r^3 \rangle$.



We know the subgroups structure of $G/N = \{N, rN, r^2N, sN, rsN, r^2sN\} \cong D_3$.

"The subgroups of the quotient G/N are the quotients of the subgroups that contain N ."

"shoeboxes; lids on"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$$\langle r \rangle \leq G$$

"shoeboxes; lids off"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$$\langle r \rangle / N \leq G/N$$

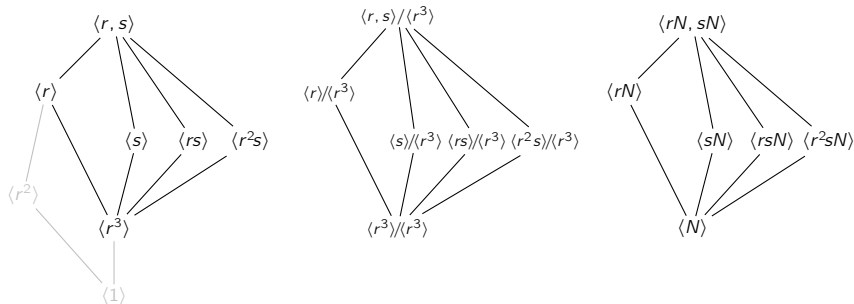
"shoes out of the box"

r^2N	r^2sN
rN	rsN
N	sN

$$\langle rN \rangle \leq G/N$$

The correspondence theorem: subgroups of quotients

Here is the subgroup lattice of $G = \text{Dic}_6$, and of the quotient G/N , where $N = \langle r^3 \rangle$.



"The subgroups of the quotient G/N are the quotients of the subgroups that contain N ."

"shoes out of the box"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$\langle s \rangle \leq G$

"shoeboxes; lids off"

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$\langle s \rangle / N \leq G/N$

"shoeboxes; lids on"

r^2N	r^2sN
rN	rsN
N	sN

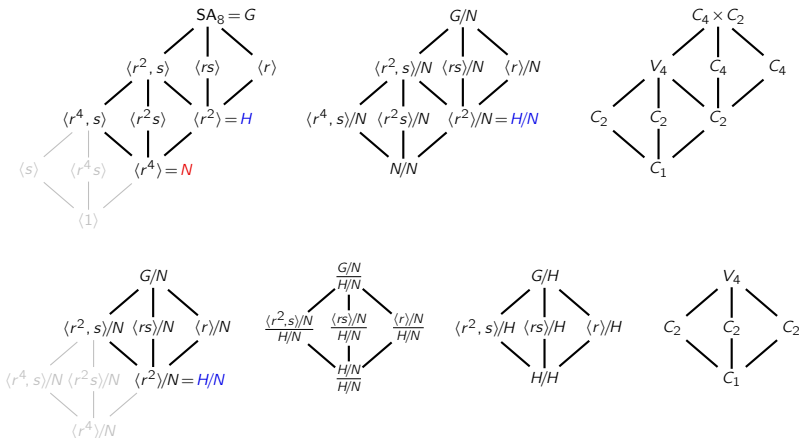
$\langle sN \rangle \leq G/N$

The fraction theorem: quotients of quotients

Fraction theorem

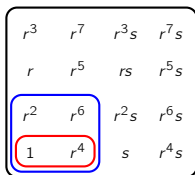
Given a chain $N \leq H \leq G$ of normal subgroups of G ,

$$(G/N)/(H/N) \cong G/H.$$

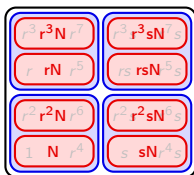


The fraction theorem: quotients of quotients

Let's continue our example of the semiabelian group $G = \text{SA}_8 = \langle r, s \rangle$.

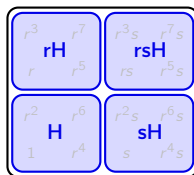


$$N \leq H \leq G$$



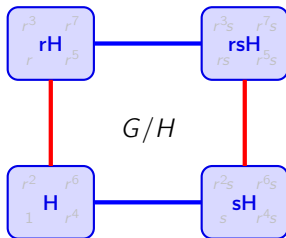
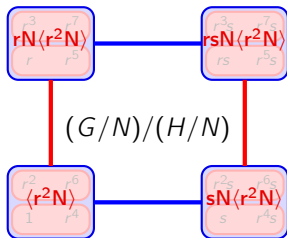
$$G/N = \langle rN, sN \rangle \cong C_4 \times C_2$$

$$H/N = \langle r^2N \rangle = \{N, r^2N\} \cong C_2$$



$$G/H = \langle rH, sH \rangle \cong V_4$$

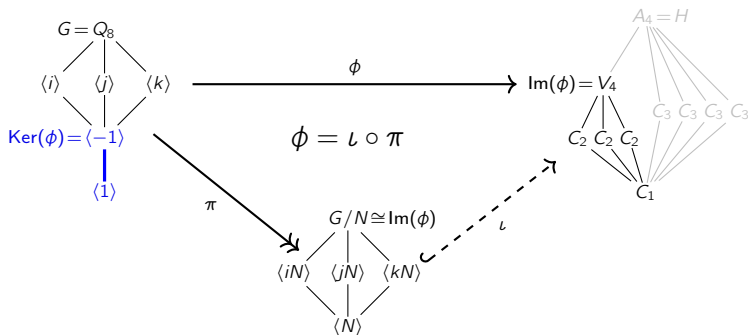
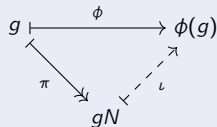
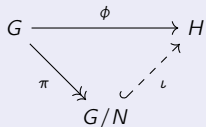
$$(G/N)/(H/N) \cong G/H$$



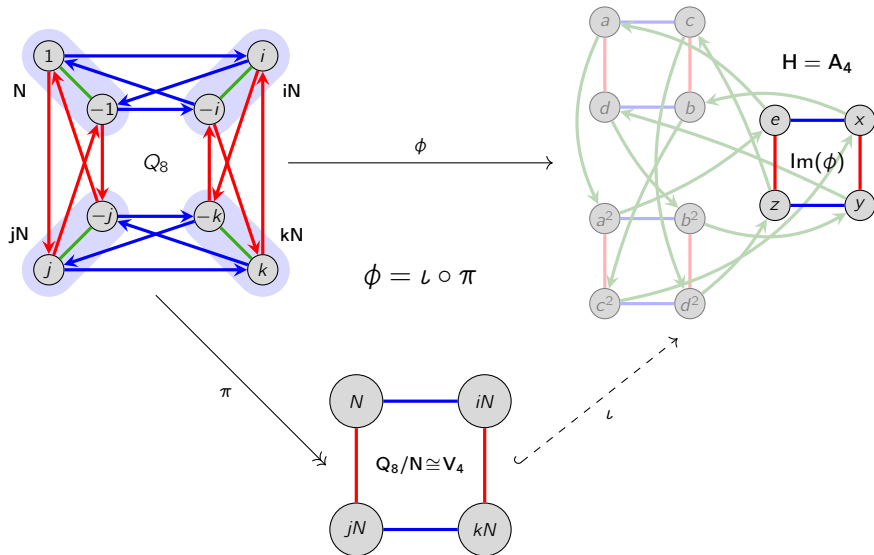
A generalization of the FHT

Theorem (exercise)

Every homomorphism $\phi: G \rightarrow H$ can be **factored** as a quotient and embedding:

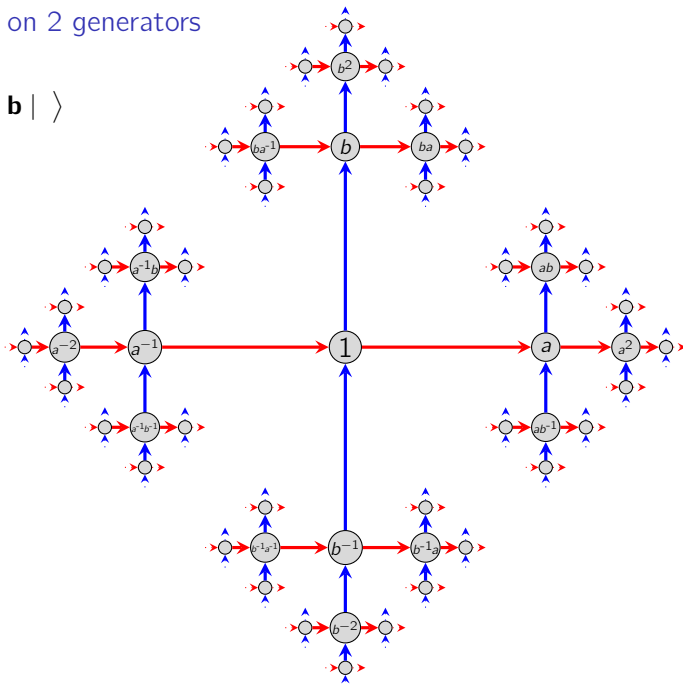


A generalization of the FHT



The free group on 2 generators

$$\mathbf{F}_2 = \langle \mathbf{a}, \mathbf{b} \mid \rangle$$



D_3 as a quotient of F_2

$$D_3 = \langle r, f \mid r^3 = r^2 = rfrf = 1 \rangle$$

