Topics: The four fundamental subspaces.

Read: Pages 1–2 of *The Four Fundamental Subspaces: 4 Lines* by Gil Strang.

`https://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf`

Do: Unless otherwise specified, let $\boldsymbol{A}$ be an $m \times n$ matrix, which we can associate with a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Let $C(\boldsymbol{A})$, $C(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$, and $N(\boldsymbol{A}^T)$ denote the column space, row space, nullspace and left nullspace, respectively.

1. For each of the following part, construct a matrix $\boldsymbol{A}$ with the specific properties specified. In this problem, $n \times 1$ column vectors are written as ordered n-tuples.
   (a) The column space contains $(1,1,1)$ and the nullspace is the line of multiples of $(1,1,1,1)$.
   (b) $\boldsymbol{A}$ is a 2×2 matrix whose column space equals its nullspace.
   (c) The nullspace of $\boldsymbol{A}$ consists of all linear combinations of $(2,2,1,0)$ and $(3,1,0,1)$. [*Hint*: Try a 2×4 matrix, and think of the "column picture."]
   (d) The nullspace of $\boldsymbol{A}$ consists of all multiples of $(4,3,2,1)$.
   (e) The column space contains $(1,1,5)$ and $(0,3,1)$ and the nullspace contains $(1,1,2)$.

2. Suppose we have a system $\boldsymbol{Ax} = \boldsymbol{b}$ with particular solution $\boldsymbol{x}_p = (2,4,0)$ and whose "homogeneous" solution $\boldsymbol{x}_n$ (i.e., the nullspace of A) is the set of scalar multiples of $(1,1,1)$.
   (a) Construct such a 2×3 system. Sketch the "grid picture."
   (b) Why can't there be a 1×3 system satisfying these conditions? Sketch the "grid picture" and show how it fails.

3. Let $\boldsymbol{A}$ be an $m \times n$ matrix.
   (a) Prove that the nullspace of $\boldsymbol{A}$ is orthogonal to the row space.
   (b) Prove that the column space of $\boldsymbol{A}$ is orthogonal to the left nullspace.