Topics: Orthogonality, Gram-Schmidt, and QR factorization

Do: Answer the following questions.

1. In this problem you will prove that orthonormal vectors are linearly independent two different ways.

   (a) Vector proof: First, suppose that $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + \cdots + c_k\boldsymbol{q}_k = \boldsymbol{0}$. Show that each $c_i = 0$. [*Hint*: Start by multipling both sides of the equation by $\boldsymbol{q}_i^T$.]

   (b) Matrix proof: Let $\boldsymbol{Q}$ be the matrix whose columns are the $\boldsymbol{q}_i$'s. Show that if $\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{x} = \boldsymbol{0}$. [*Hint*: Since $\boldsymbol{Q}$ need not be square, you cannot assume $\boldsymbol{Q}^{-1}$ exists, but $\boldsymbol{Q}^T$ of course will.]

2. Recall that if $||\boldsymbol{u}|| = 1$, then the rank-1 matrix $\boldsymbol{u}\boldsymbol{u}^T$ is the projection matrix onto $\boldsymbol{u}$. In this case, $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T$ is a *reflection matrix*.

   (a) Reflecting twice across the same axis is the identity. Verify that indeed, $\boldsymbol{Q}^2 = \boldsymbol{I}$.

   (b) Compute $\boldsymbol{Q}\boldsymbol{u}$, and simplify this expression as much as possible.

   (c) Suppose $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}$. Compute $\boldsymbol{Q}\boldsymbol{v}$, and simplify as much as possible.

   (d) Describe in plain English which subspace $\boldsymbol{Q}$ is reflecting across. Your answer should involve $\boldsymbol{u}$. Include a sketch.

   (e) Compute the reflection matrix $\boldsymbol{Q}_1 = \boldsymbol{I} - 2\boldsymbol{u}_1\boldsymbol{u}_1^T$ where $\boldsymbol{u}_1 = (0, 1)$. Compute $\boldsymbol{Q}_1\boldsymbol{x}_1$, where $\boldsymbol{x}_1 = (a, b)$, and sketch the vectors $\boldsymbol{u}_1$, $\boldsymbol{x}_1$, and $\boldsymbol{Q}_1\boldsymbol{x}_1$ in the plane.

   (f) Compute the reflection matrix $\boldsymbol{Q}_2 = \boldsymbol{I} - 2\boldsymbol{u}_2\boldsymbol{u}_2^T$ where $\boldsymbol{u}_2 = (0, \sqrt{2}/2, \sqrt{2}/2)$. Compute $\boldsymbol{Q}_2\boldsymbol{x}_2$, where $\boldsymbol{x}_2 = (1, 1, 1)$, and sketch the vectors $\boldsymbol{u}_2$, $\boldsymbol{x}_2$, and $\boldsymbol{Q}_2\boldsymbol{x}_2$ in $\mathbb{R}^3$.

3. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be the (independent) column vectors of the matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

   whose column space is the orthogonal complement to the vector $(1, 1, 1, 1)$. Use the Gram-Schmidt process to produce an orthonormal basis $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$. Then write $\boldsymbol{M} = \boldsymbol{Q}\boldsymbol{R}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is upper-triangular.