Topics: Vector spaces and subspaces

Do: Answer the following questions.

1. Let X be a vector space over a field K. Let 0 be the zero element of K, and $\mathbf{0}$ the zero element of X. Using only the definitions of a group, vector space, and field, carefully prove the following.
   (i) $(-1)x = -x$ for all $x \in X$.
   (ii) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

2. Let X be a vector space.
   (i) Prove that for any collection $\big\{Y_\alpha \mid \alpha \in A\big\}$ of subspaces, $\bigcap_\alpha Y_\alpha$ is a subspace.
   (ii) Prove that for any subset $S \subseteq X$ of a vector space,
   $$\mathrm{Span}(S) = \bigcap_{S \subseteq Y_\alpha \leq X} Y_\alpha.$$
   Here, the intersection is taken over all subspaces Y_α of X that contain S.

3. Let Y and Z be subspaces of a vector space X, with $Y + Z = X$. That is,
   $$Y + Z := \big\{y + z \mid y \in Y,\, z \in Z\big\} = X.$$
   Prove that the following are equivalent:
   (i) $Y \cap Z = \{0\}$.
   (ii) Every $x \in X$ can be written *uniquely* as $x = y + z$, for some $y \in Y$ and $z \in Z$.