Topics: The isomorphism theorems

Do: Answer the following questions.

1. Let $f\colon X \to U$ be a linear map between vector spaces. In this problem, you will prove the *first isomorphism theorem*, which says that $X/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$, where $\operatorname{Im}(f) = f(X)$.

   (a) Let $Z = \operatorname{Ker}(f)$, and $x+Z = \{x+z \mid z \in Z\} = \overline{x}$ be the equivalence class containing x; either notation is fine, feel free to use either. Define a linear map

   $$\iota\colon X/Z \longrightarrow \operatorname{Im}(f), \qquad x+Z \longmapsto f(x).$$

   Carefully give a formal definition of what it means for ι to be well-defined, and then prove that it is.

   (b) Prove that ι is linear, one-to-one, and onto, and conclude that $X/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$.

2. In this problem, you will prove the main part of the second isomorphism theorem, sometimes called the *correspondence theorem*. Let Z be a subspace of X.

   (a) Show that if $Z \le Y \le X$, then Y/Z is a subspace of X/Z.

   (b) Show that *every* subspace of X/Z arises in this manner.

   In this problem, it is easier to use "coset notation" $x+Z$ instead of $\overline{x}$, because you may need to speak of both equivalence classes $x+Y$ and $x+Z$.

3. Prove the *third isomorphism theorem*: if $Z \le Y \le X$, then $(X/Z)/(Y/Z) \cong X/Y$. [*Hint*: Define a map $X/Z \to X/Y$, show it is linear, onto, and has kernel Y/Z. Then apply the first isomorphism theorem.]