Topics: The transpose of a linear map

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K.

1. Recall that the *transpose* of a linear map $T\colon X \to U$ is the linear map $T\colon U' \to X'$ between the dual spaces such that $(\ell, Tx) = (T'\ell, x)$ for all $x \in X$ and $\ell \in U'$. In class, we showed that $(ST)' = T'S'$. Show the following.

   (a) $(S+T)' = S' + T'$, for $S, T\colon X \to U$.

   (b) $I' = I$, where $I\colon X \to X$ is the identity map.

   (c) $(T^{-1})' = (T')^{-1}$.

2. Let $T\colon X \to U$ be a linear map with range R_T and nullspace N_T. Recall that the *annihilator* of a subspace $Y \leq X$ is

$$Y^\perp = \big\{\ell \in X' \colon (\ell, y) = 0,\, \forall y \in Y\big\}.$$

   In class, we showed that $R_T^\perp = N_{T'}$. As an immediate corollary, applying this statement to $T'\colon U' \to X'$ gives the analogous result $R_{T'}^\perp = N_T$. Give an alternate direct proof of this fact, akin to what we did in class for $R_T^\perp = N_{T'}$.