Topic: The matrix of a linear map

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K.

1. Let $\mathcal{P}_n$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than n.

   (a) Show that the map $T\colon \mathcal{P}_3 \to \mathcal{P}_4$ given by

   $$T(p(x)) = 6\int_1^x p(t)\,dt$$

   is linear. Determine whether it is 1–1 or onto.

   (b) Let $\mathcal{B}_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $\mathcal{B}_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

2. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show how to construct bases $\mathcal{B}_X$ for X and $\mathcal{B}_U$ for U such that the matrix of T in block form is

   $$M = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

   where I_r is the $r \times r$ identity matrix, and the other blocks are either empty or contain all zeros. Prove/justify all of your claims.

3. Find an explicit counterexample to the following statement: If the matrix of a linear map $T\colon X \to X$ is A, with respect to an input basis $x_1, \dots, x_n$ and output basis $u_1, \dots, u_n$, then the matrix of T^2 is A^2.