Topic: Multilinearity

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K. A k-linear form $f\colon X^k \to K$ is:

- *symmetric* if $f(x_1,\dots,x_k) = \pi \cdot f(x_1,\dots,x_k) := f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(k)})$ for all $\pi \in S_k$,
- *skew-symmetric* if $\tau \cdot f(x_1,\dots,x_k) = -f(x_1,\dots,x_k)$ for all transpositions $\tau = (ij) \in S_k$,
- *alternating* if $f(x_1,\dots,x_k) = 0$ whenever $x_i = x_j$.

1. Let f be a bilinear form over a vector space X with basis $\{x_1, x_2\}$.
   (a) Assume f is alternating. Determine a formula for $f(u,v)$ in terms of each $f(x_i,x_j)$ and the coefficients used to express u and v with this basis.
   (b) Repeat Part (a) but assume that f is symmetric, and that $f(x,x) = 0$ for all $x \in X$.
   (c) Repeat Part (a) but now assume that X is 3-dimensional, with basis $\{x_1, x_2, x_3\}$.

2. Let $A = (c_1,\dots,c_n)$ be an $n \times n$ matrix (c_i is a column vector), and let B be the matrix obtained from A by adding k times the i^{th} column of A to the j^{th} column of A, for some $i \neq j$. Show that $\det A = \det B$. You may assume that the determinant is an alternating n-linear form.

3. Let f be an alternating k-linear form. Show that if $y_1,\dots,y_k$ are linearly dependent, then $f(y_1,\dots,y_k) = 0$. Then give an explicit counterexample to show how the converse fails.