Topic: Eigenspaces and Jordan canonical form.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

1. Consider the matrices $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
   (a) Decompose $\mathbb{R}^3$ into a direct sum of eigenspaces of each matrix.
   (b) Further decompose the 2-dimensional A-eigenspace as a direct sum of two 1-dimensonal B-eigenspaces, and vice-versa.
   (c) Write $\mathbb{R}^3$ as a direct sum of three 1-dimensional subspaces that are common eigenspaces of A and B, two different ways.
   (d) For each of your answers to Part (c), find a matrix P so that $P^{-1}AP = D_A$ and $P^{-1}BP = D_B$, where D_A and D_B are diagonal.
2. Let $\boldsymbol{A}$ be a 7×7 matrix over $\mathbb{C}$ with minimal polynomial $m(t) = (t-1)^3(t-2)^2$.
   (a) List all possible Jordan canonical forms of A up to similarity.
   (b) For each matrix from Part (a), find the rank of $(\boldsymbol{A} - \boldsymbol{I})^k$ and $(\boldsymbol{A} - 2\boldsymbol{I})^k$, for $k \in \mathbb{N}$.
3. Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix $\boldsymbol{A}$ is *nilpotent* if $\boldsymbol{A}^k = 0$ for some $k \in \mathbb{N}$, and $\boldsymbol{A}$ is *idempotent* if $\boldsymbol{A}^2 = \boldsymbol{A}$.
   (a) Prove that if $\boldsymbol{A}^k = \boldsymbol{A}$ for some integer $k > 1$, then $\boldsymbol{A}$ is diagonalizable.
   (b) Prove that idempotent matrices are similar if and only if they have the same trace.
   (c) Prove that if $\boldsymbol{A}$ is nilpotent, then $A^n = 0$.
   (d) Prove that if $\boldsymbol{A}$ is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine A up to similarity.
   (e) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_{\boldsymbol{A}} = \dim N_{\boldsymbol{B}}$. Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar. Show by example that this can fail for 7×7 matrices.