Topic: Inner products.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{R}$.

1. Let $X = \mathbb{R}^3$, and define the inner product by

$$\langle x, y \rangle = y^T A x = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Find the norm of the three unit basis vectors e_1, e_2, and e_3, the angles between them, and the orthogonal complements of the lines that they span.

2. Given a linear map $A \colon X \to X$, the function $f \colon X \to X$ defined by $f(x, y) = x^T A y$ is an inner product on X if A is symmetric and positive-definite.

   (a) Find the matrix A that defines the following inner product:

$$f(x, y) = 3x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - x_2y_3 - x_3y_2 + 3x_3y_3 = x^T A y.$$

   (b) Find an orthonormal basis v_1, v_2, v_3 of $\mathbb{R}^3$ so that with respect to this basis, $f(z, w) = z^T D w$ for some diagonal matrix D.

   (c) Write a formula for $f(z, w)$ like in Part (b), but with respect to this new basis.

3. Let f and g be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

   (a) $\left(\int_0^1 f(t)g(t)\, dt \right)^2 \leq \int_0^1 f(t)^2\, dt \int_0^1 g(t)^2\, dt$

   (b) $\left(\int_0^1 (f(t) + g(t))^2\, dt \right)^{1/2} \leq \left(\int_0^1 f(t)^2\, dt \right)^{1/2} + \left(\int_0^1 g(t)^2 dt \right)^{1/2}.$