Topic: Self-adjoint matrices, quadratic forms, left & right inverses.

Read: *Spectral theorems and singular value decomposition (SVD)*, by Shuhong Gao.

Do: Answer the following question.

1. Consider the quadratic form $q(x) = 2x_1^2 + 6x_1x_2 + 2x_2^2$.

   (a) Write this as $q(x) = x^T Ax$, for some A.

   (b) Write $A = PDP^T$, where D is a diagonal matrix and P is orthogonal with determinant 1, i.e., a rotation matrix.

   (c) Change variables by letting $z = P^T x$. Sketch the level curve $q(x) = 1$ in both the z_1z_2-plane and in the x_1x_2-plane.

2. Let $A\colon X \to U$ be a linear map between finite-dimensional vector spaces, with $\dim X = n$ and $\dim U = m$. The map A has a *left inverse* if there is a linear map $L\colon U \to X$ such that $LA = I_X$, the identity on X. It has a *right inverse* if there is a linear map $R\colon U \to X$ such that $AR = I_U$ is the identity on U.

   (a) Show that if A has a left inverse, then $Ax = u$ has *at most* one solution. Give a condition on u that completely characterizes when there is a solution.

   (b) Show that if A has a right inverse, then $Ax = u$ has *at least* one solution. If $Ax_p = u$ for some particular $x_p \in X$, then describe all solutions for x in this case. What condition ensures that there will be *only* one solution?

   (c) What are the possibilities for the rank of A if it has a left inverse? What if it has a right inverse?