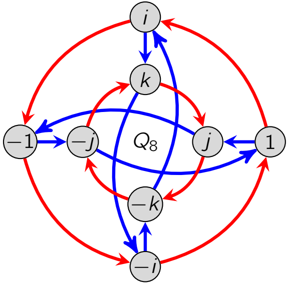
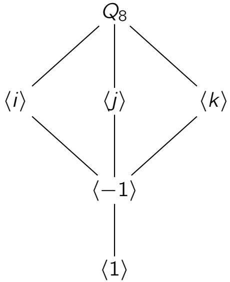
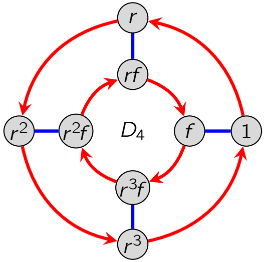
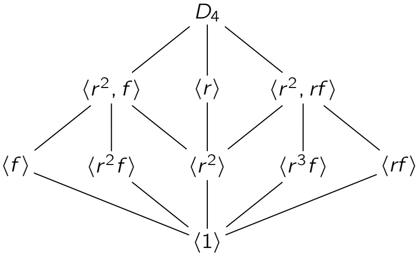
# Two nonabelian groups of order 8



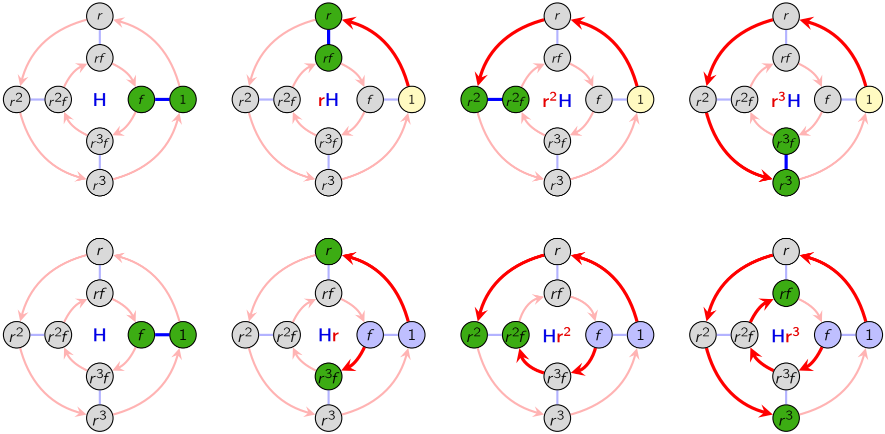



# Left vs. right cosets

## Definition

Let $H \leq G$. Given $x \in G$, its left coset $xH$ and right coset $Hx$ are:

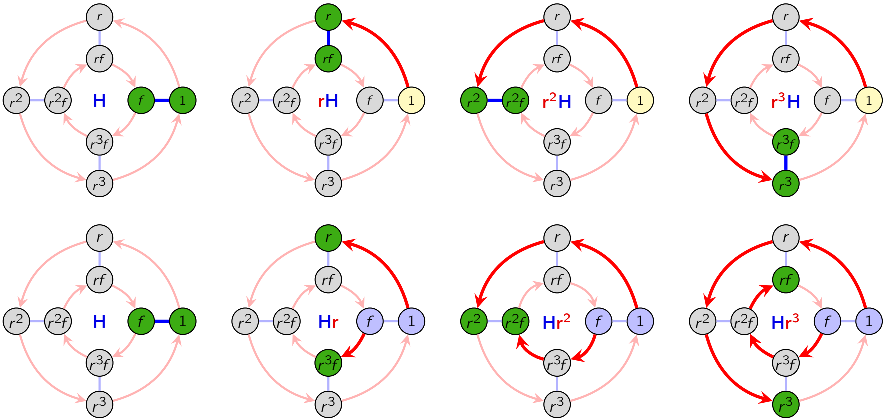
$$xH = \{xh \mid h \in H\}, \qquad Hx = \{hx \mid h \in H\}.$$



# Left vs. right cosets

| H | $r^2H$ | $rH$ | $r^3H$ |
|---|---|---|---|
| $f$ | $r^2f$ | $rf$ | $r^3$ |
| $1$ | $r^2$ | $r$ | $r^3f$ |

| H | $Hr^2$ | | | |
|---|---|---|---|---|
| $f$ | $fr^2$ | $fr^3$ | $r^3$ | $Hr^3$ |
| $1$ | $r^2$ | $r$ | $fr$ | $Hr$ |

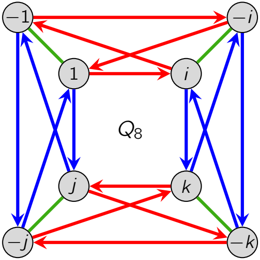


# Quotients

## Key idea

The quotient of $G$ by a subgroup $H$ exists when the (left) cosets of $H$ form a group.

Here is the quotient of $G = Q_8$ by the subgroup $H = \langle -1 \rangle = \{1, -1\}$.




| | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ | $k$ | $-k$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ | $k$ | $-k$ |
| $-1$ | $-1$ | $1$ | $-i$ | $i$ | $-j$ | $j$ | $-k$ | $k$ |
| $i$ | $i$ | $-i$ | $-1$ | $1$ | $k$ | $-k$ | $-j$ | $j$ |
| $-i$ | $-i$ | $i$ | $1$ | $-1$ | $-k$ | $k$ | $j$ | $-j$ |
| $j$ | $j$ | $-j$ | $-k$ | $k$ | $-1$ | $1$ | $i$ | $-i$ |
| $-j$ | $-j$ | $j$ | $k$ | $-k$ | $1$ | $-1$ | $-i$ | $i$ |
| $k$ | $k$ | $-k$ | $j$ | $-j$ | $-i$ | $i$ | $-1$ | $1$ |
| $-k$ | $-k$ | $k$ | $-j$ | $j$ | $i$ | $-i$ | $1$ | $-1$ |

$Q_8/\langle -1 \rangle \cong V_4$

| | $\pm 1$ | $\pm i$ | $\pm j$ | $\pm k$ |
|---|---|---|---|---|
| $\pm 1$ | $\pm 1$ | $\pm i$ | $\pm j$ | $\pm k$ |
| $\pm i$ | $\pm i$ | $\pm 1$ | $\pm k$ | $\pm j$ |
| $\pm j$ | $\pm j$ | $\pm k$ | $\pm 1$ | $\pm i$ |
| $\pm k$ | $\pm k$ | $\pm j$ | $\pm i$ | $\pm 1$ |

# Quotients

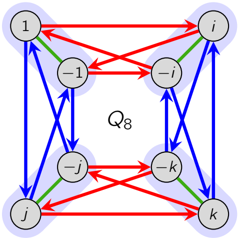
Denote the set of left cosets of $H$ in $G$ by

$$G/H := \{xH \mid x \in G\}.$$

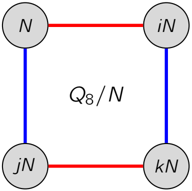
## Key idea

The quotient of $G$ by a subgroup $H$ exists when the (left) cosets of $H$ form a group.

This is well-defined precisely when $H$ is normal. (Left and right cosets coincide.)



Cluster the left cosets of $N$

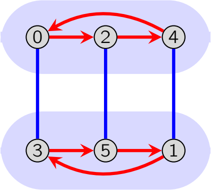


Collapse cosets into single nodes

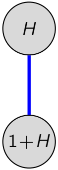
| | $N$ | $iN$ | $jN$ | $kN$ |
|---|---|---|---|---|
| $N$ | $N$ | $iN$ | $jN$ | $kN$ |
| $iN$ | $iN$ | $N$ | $kN$ | $jN$ |
| $jN$ | $jN$ | $kN$ | $N$ | $iN$ |
| $kN$ | $kN$ | $jN$ | $iN$ | $N$ |

Elements of the quotient are cosets of $N$

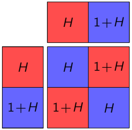
# Quotients



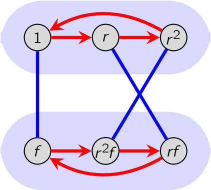

Cluster the
left cosets of $H \leq \mathbb{Z}_6$



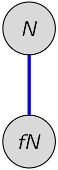

Collapse cosets
into single nodes



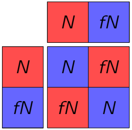

Elements of the quotient
are cosets of $H$




Cluster the
left cosets of $N \leq D_3$



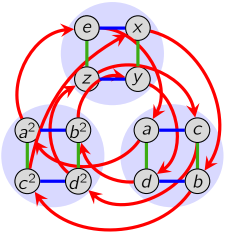

Collapse cosets
into single nodes

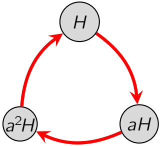



Elements of the quotient
are cosets of $N$

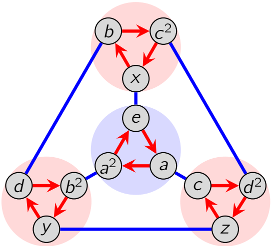
# Quotients

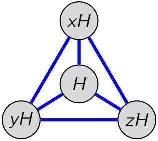
Let's revisit $N = \langle (12)(34), (13)(24) \rangle$ and $H = \langle (123) \rangle$ of $A_4$.



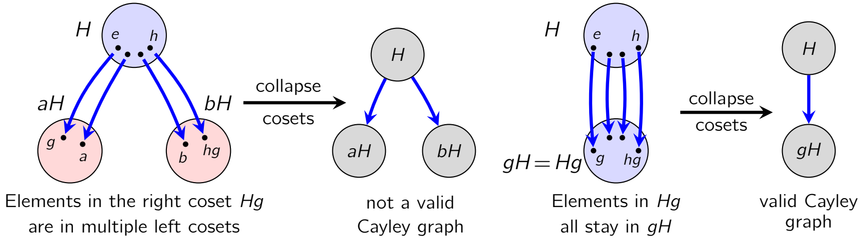

| | $H$ | $aH$ | $a^2H$ |
|---|---|---|---|
| $H$ | $H$ | $aH$ | $a^2H$ |
| $aH$ | $aH$ | $a^2H$ | $H$ |
| $a^2H$ | $a^2H$ | $H$ | $aH$ |



# When and why the quotient process works

In the following: *the right coset Hg are the nodes at the "arrowtips"*.




## Key idea

If $H$ is normal subgroup of $G$, then the quotient group $G/H$ exists.

If $H$ is not normal, then following the blue arrows from $H$ is ambiguous.

In other words, it depends on our where we start within $H$.

# What does it mean to “multiply” two cosets?

## Quotient theorem

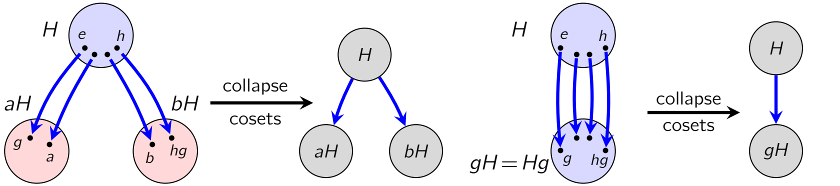
If $H \trianglelefteq G$, the set of cosets $G/H$ forms a group, with binary operation

$$aH \cdot bH := abH.$$

It is clear that $G/H$ is closed under this operation.

We have to show that this operation is well-defined.

By that, we mean that it *does not depend on our choice of coset representative*.




our destination depends on where in $H$ we start

the quotient process does not yield a group

destination doesn’t depend on where we start in $H$

quotient process succeeds

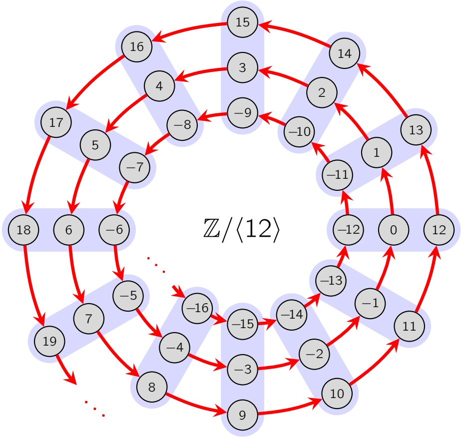
# A familiar example

Consider the subgroup $H = \langle 12 \rangle = 12\mathbb{Z}$ of $G = \mathbb{Z}$.

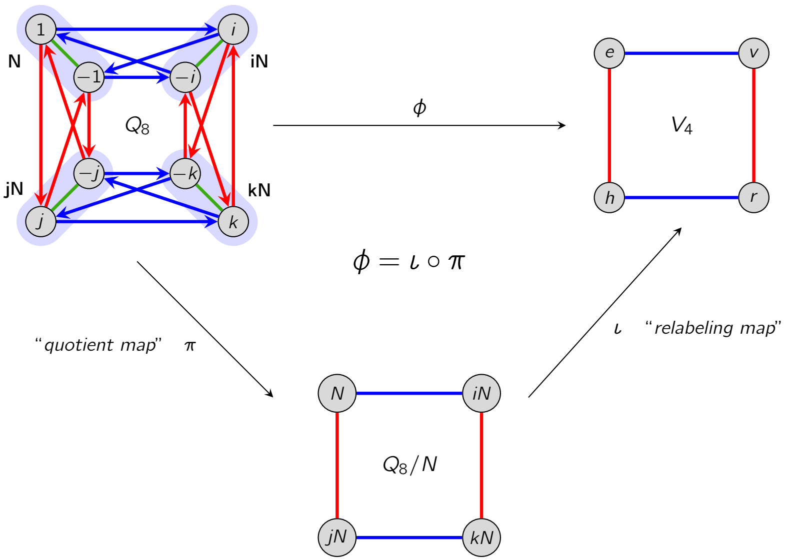
The cosets of $H$ are the congruence classes modulo 12.

Since this group is additive, the condition $aH \cdot bH$ becomes $(a+H)+(b+H) = a+b+H$:

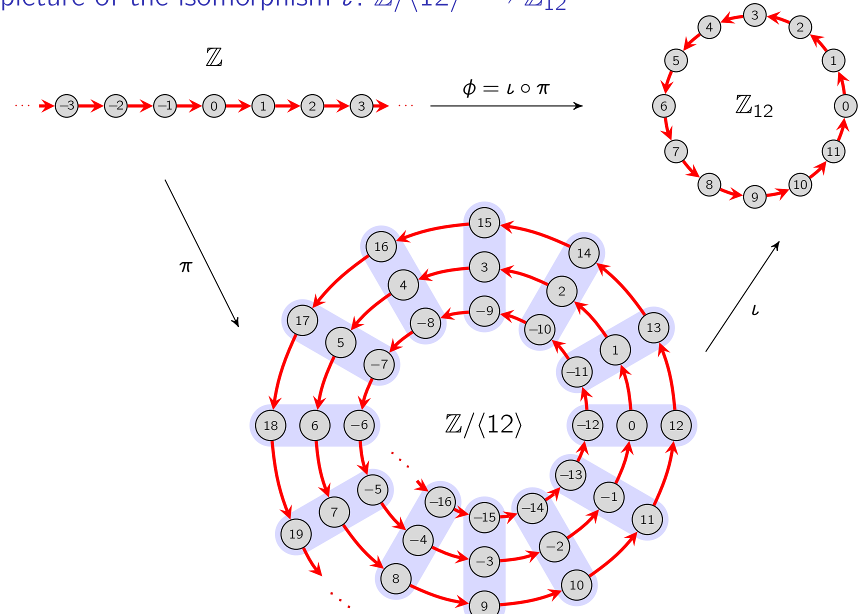
*"(the coset containing a) + (the coset containing b) = the coset containing $a+b$."*



# Visualizing the FHT via Cayley graphs



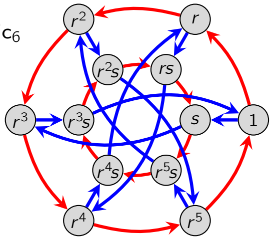
# A picture of the isomorphism $\iota\colon \mathbb{Z}/\langle 12\rangle \longrightarrow \mathbb{Z}_{12}$

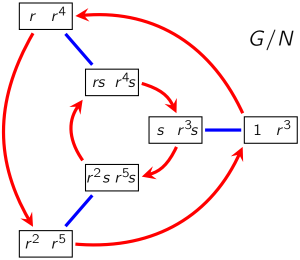


# The correspondence theorem: subgroups of quotients

Compare $G = \mathrm{Dic}_6$ with the quotient by $N = \langle r^3 \rangle$.

$G = \mathrm{Dic}_6$




We know the subgroups structure of $G/N = \{N,\ rN,\ r^2N,\ sN,\ rsN,\ r^2sN\} \cong D_3$.

"*The subgroups of the quotient $G/N$ are the quotients of the subgroups that contain $N$.*"

"*shoeboxes; lids on*"

| | | | |
|---|---|---|---|
| $r^2$ | $r^5$ | $r^2s$ | $r^5s$ |
| $r$ | $r^4$ | $rs$ | $r^4s$ |
| $1$ | $r^3$ | $s$ | $r^3s$ |

$\langle r \rangle \leq G$

"*shoeboxes; lids off*"

| | | | |
|---|---|---|---|
| $r^2$ | $r^5$ | $r^2s$ | $r^5s$ |
| $r$ | $r^4$ | $rs$ | $r^4s$ |
| $1$ | $r^3$ | $s$ | $r^3s$ |

$\langle r \rangle / N \leq G/N$

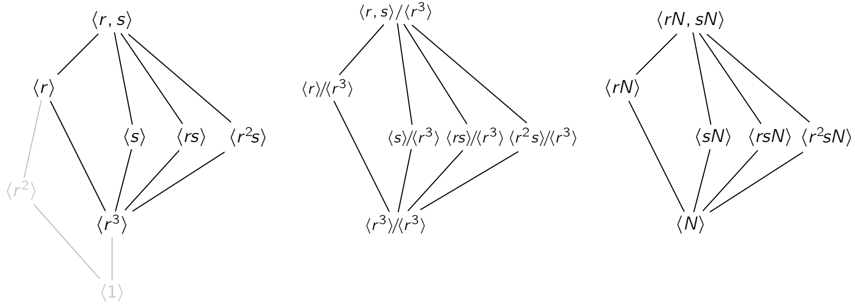
"*shoes out of the box*"

| | |
|---|---|
| $r^2N$ | $r^2sN$ |
| $rN$ | $rsN$ |
| $N$ | $sN$ |

$\langle rN \rangle \leq G/N$

# The correspondence theorem: subgroups of quotients

Here is the subgroup lattice of $G = \mathrm{Dic}_6$, and of the quotient $G/N$, where $N = \langle r^3 \rangle$.




"*The subgroups of the quotient $G/N$ are the quotients of the subgroups that contain $N$.*"

"*shoes out of the box*"

| | | | |
|---|---|---|---|
| $r^2$ | $r^5$ | $r^2s$ | $r^5s$ |
| $r$ | $r^4$ | $rs$ | $r^4s$ |
| $1$ | $r^3$ | $s$ | $r^3s$ |

$\langle s \rangle \leq G$

"*shoeboxes; lids off*"

| | | | |
|---|---|---|---|
| $r^2$ | $r^5$ | $r^2s$ | $r^5s$ |
| $r$ | $r^4$ | $rs$ | $r^4s$ |
| $1$ | $r^3$ | $s$ | $r^3s$ |

$\langle s \rangle / N \leq G/N$

"*shoeboxes; lids on*"

| | |
|---|---|
| $r^2N$ | $r^2sN$ |
| $rN$ | $rsN$ |
| $N$ | $sN$ |

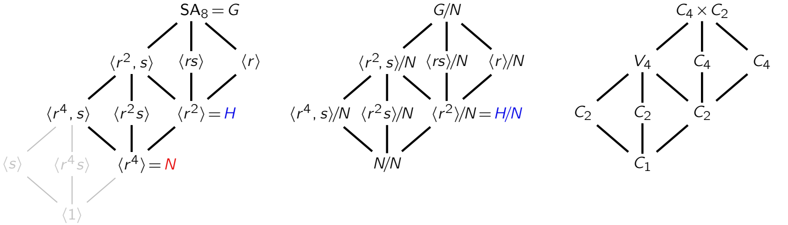
$\langle sN \rangle \leq G/N$

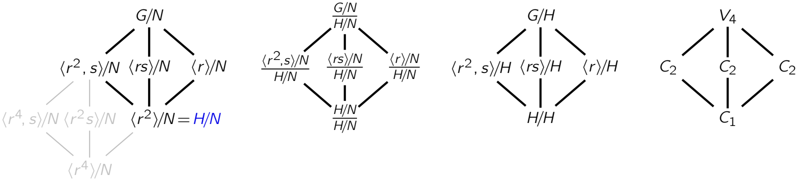
# The fraction theorem: quotients of quotients

## Fraction theorem

Given a chain $N \leq H \leq G$ of normal subgroups of $G$,

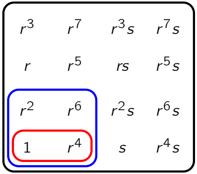
$$(G/N)/(H/N) \cong G/H.$$



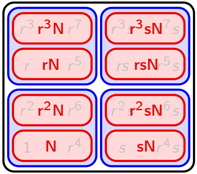



# The fraction theorem: quotients of quotients

Let's continue our example of the semiabelian group $G = \mathrm{SA}_8 = \langle r, s \rangle$.

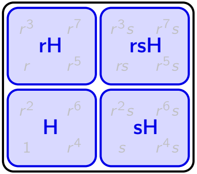



$N \leq H \leq G$



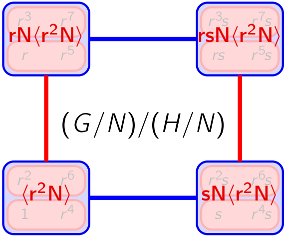

$G/N = \langle rN, sN \rangle \cong C_4 \times C_2$

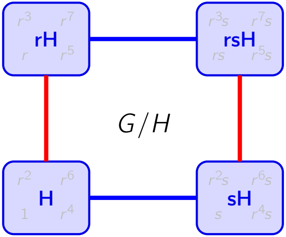
$H/N = \langle r^2N \rangle = \{N, r^2N\} \cong C_2$




$G/H = \langle rH, sH \rangle \cong V_4$

$(G/N)/(H/N) \cong G/H$

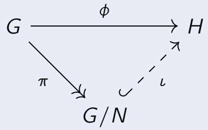


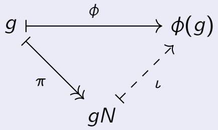



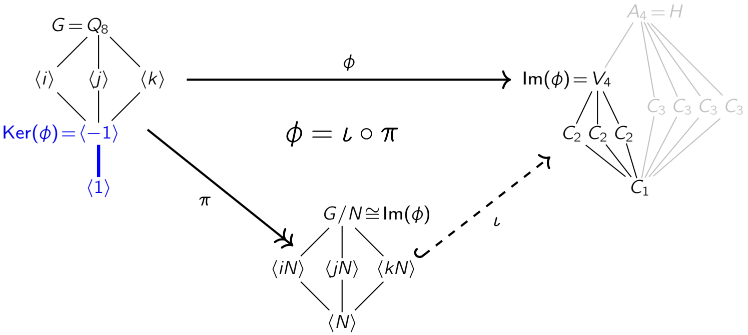
# A generalization of the FHT

## Theorem (exercise)

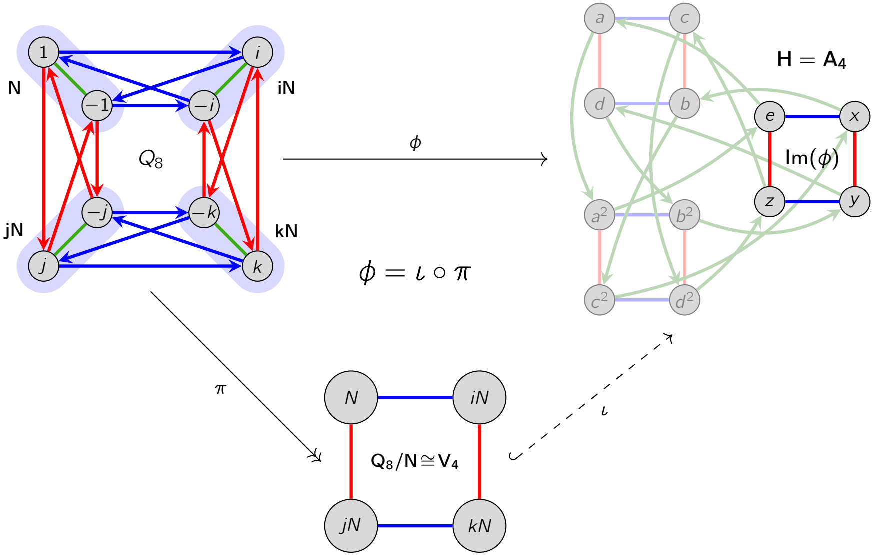
Every homomorphism $\phi \colon G \to H$ can be factored as a quotient and embedding:



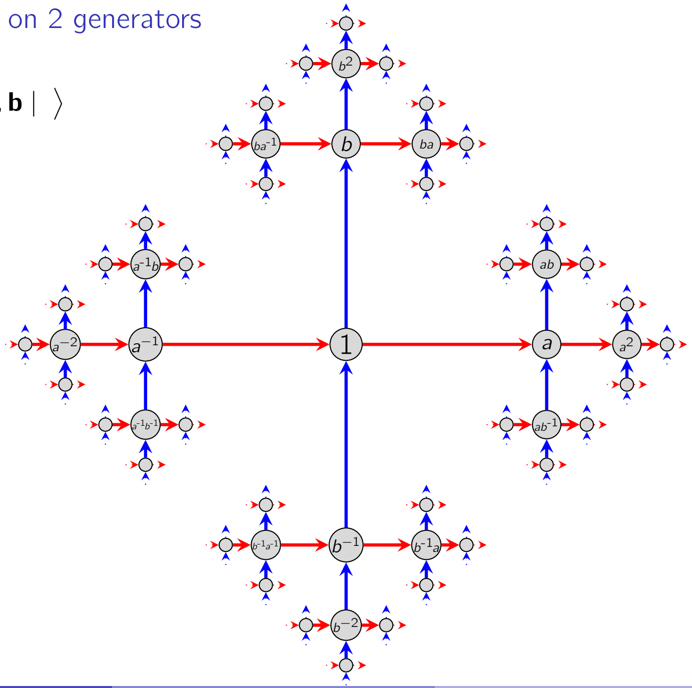



# A generalization of the FHT



# The free group on 2 generators

$$\mathbf{F_2} = \langle \mathbf{a}, \mathbf{b} \mid \; \rangle$$



# $D_3$ as a quotient of $F_2$

$$\mathbf{D_3} = \langle \mathbf{r}, \mathbf{f} \mid \mathbf{r}^3 = \mathbf{r}^2 = \mathbf{rfrf} = \mathbf{1} \rangle$$