Topics: The four fundamental subspaces.

Read: Pages 1–2 of *The Four Fundamental Subspaces: 4 Lines* by Gil Strang.

`https://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf`

Do: Unless otherwise specified, let $\mathbf{A}$ be an $m \times n$ matrix, which we can associate with a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Let $C(\mathbf{A})$, $C(\mathbf{A}^T)$, $N(\mathbf{A})$, and $N(\mathbf{A}^T)$ denote the column space, row space, nullspace and left nullspace, respectively.

Problem 1. For each of the following part, construct a matrix $\mathbf{A}$ with the specific properties specified. In this problem, $n \times 1$ column vectors are written as ordered n-tuples.

1. The column space contains $(1,1,1)$ and the nullspace is the line of multiples of $(1,1,1,1)$.
2. $\mathbf{A}$ is a 2×2 matrix whose column space equals its nullspace.
3. The nullspace of $\mathbf{A}$ consists of all linear combinations of $(2,2,1,0)$ and $(3,1,0,1)$. [*Hint*: Try a 2×4 matrix, and think of the "column picture."]
4. The nullspace of $\mathbf{A}$ consists of all multiples of $(4,3,2,1)$.
5. The column space contains $(1,1,5)$ and $(0,3,1)$ and the nullspace contains $(1,1,2)$.

Problem 2. Suppose we have a system $\mathbf{Ax} = \mathbf{b}$ with particular solution $\mathbf{x}_p = (2,4,0)$ and whose "homogeneous" solution $\mathbf{x}_n$ (i.e., the nullspace of A) is the set of scalar multiples of $(1,1,1)$.

1. Construct such a 2×3 system. Sketch the "grid picture."
2. Why can't there be a 1×3 system satisfying these conditions? Sketch the "grid picture" and show how it fails.

Problem 3. Let $\mathbf{A}$ be an $m \times n$ matrix.

1. Prove that the nullspace of $\mathbf{A}$ is orthogonal to the row space.
2. Prove that the column space of $\mathbf{A}$ is orthogonal to the left nullspace.