

Topics: Projection and least squares.

Do: Answer the following questions, and feel free to use a computer to solve *any* systems of equations you encounter throughout this problem.

Problem 1.

Consider the four data points $(t_i, b_i) = (0, 0), (1, 8), (3, 8),$ and $(4, 20)$. Let $\mathbf{t} = (0, 1, 3, 4)$ be the vector of inputs and $\mathbf{b} = (0, 8, 8, 20)$ the vector of outputs.

- (i) If there were a straight line $b = C + Dt$ through these four points, then a certain equation $\mathbf{Ax} = \mathbf{b}$ would have a solution, where $\mathbf{x} = (C, D)$. Write this equation in matrix form (that is, find $\mathbf{A}$).
- (ii) Instead, we wish to find the “best fit” line, which means we need to solve $\mathbf{Ax} = \mathbf{p}$, where $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$. Write down the *normal equations* $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$, where $\hat{\mathbf{x}} = (\hat{C}, \hat{D})$, and solve for $\hat{\mathbf{x}}$.
- (iii) Check that $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is orthogonal to both columns of $\mathbf{A}$. Compute $\|\mathbf{e}\|$, which is the shortest distance from $\mathbf{b}$ to the column space of $\mathbf{A}$. Sketch a diagram of $\mathbf{e}$, $\mathbf{b}$, $\mathbf{p}$, and the orthogonal subspaces $C(\mathbf{A})$ and $N(\mathbf{A}^T)$ to illustrate this.
- (iv) Plot the four data points in $\mathbb{R}^2$ (on the tb -plane) and sketch the best fit line through them that you just found. Clearly mark what the vectors $\mathbf{b} = (b_1, b_2, b_3, b_4)$, $\mathbf{e} = (e_1, e_2, e_3, e_4)$, and $\mathbf{p} = (p_1, p_2, p_3, p_4)$ represent.
- (v) Write down $E := \|\mathbf{Ax} - \mathbf{b}\|^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$, and compute $\partial E / \partial C$ and $\partial E / \partial D$. Set these derivatives equal to zero, and obtain scalars of the normal equations $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$.
- (vi) The method above found the best fit degree-1 polynomial (line). Now, find the best fit degree-0 polynomial (horizontal line) $b = C$. Note that this will be a 4×1 system instead of a 4×2 system. Compute the vectors $\mathbf{p}$ and $\mathbf{e}$, and the (squared) error $\|\mathbf{e}\|^2$.
- (vii) Find the best fit parabola (degree-2 polynomial) $b = C + Dt + Et^2$. On a new set of axes, plot the four data points and this parabola. Compute the vectors $\mathbf{p}$ and $\mathbf{e}$, and the (squared) error $\|\mathbf{e}\|^2$.
- (viii) Find the best fit cubic (degree-3 polynomial) $b = C + Dt + Et^2 + Ft^3$. On a new set of axes, plot the four data points and this cubic. Compute the vectors $\mathbf{p}$ and $\mathbf{e}$, and the (squared) error $\|\mathbf{e}\|^2$.