Topics: Orthogonality, Gram-Schmidt, and QR factorization

Do: Answer the following questions.

Problem 1. In this problem you will prove that orthonormal vectors are linearly independent two different ways.

(i) Vector proof: First, suppose that $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_k\mathbf{q}_k = \mathbf{0}$. Show that each $c_i = 0$. [*Hint*: Start by multiplying both sides of the equation by $\mathbf{q}_i^T$.]

(ii) Matrix proof: Let $\mathbf{Q}$ be the matrix whose columns are the $\mathbf{q}_i$'s. Show that if $\mathbf{Qx} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. [*Hint*: Since $\mathbf{Q}$ need not be square, you cannot assume $\mathbf{Q}^{-1}$ exists, but $\mathbf{Q}^T$ of course will.]

Problem 2. Recall that if $||\mathbf{u}|| = 1$, then the rank-1 matrix $\mathbf{uu}^T$ is the projection matrix onto $\mathbf{u}$. In this case, $\mathbf{Q} = \mathbf{I} - 2\mathbf{uu}^T$ is a *reflection matrix*.

(i) Reflecting twice across the same axis is the identity. Verify that indeed, $\mathbf{Q}^2 = \mathbf{I}$.

(ii) Compute $\mathbf{Qu}$, and simplify this expression as much as possible.

(iii) Suppose $\mathbf{v}$ is orthogonal to $\mathbf{u}$. Compute $\mathbf{Qv}$, and simplify as much as possible.

(iv) Describe in plain English which subspace $\mathbf{Q}$ is reflecting across. Your answer should involve $\mathbf{u}$. Include a sketch.

(v) Compute the reflection matrix $\mathbf{Q}_1 = \mathbf{I} - 2\mathbf{u}_1\mathbf{u}_1^T$ where $\mathbf{u}_1 = (0, 1)$. Compute $\mathbf{Q}_1\mathbf{x}_1$, where $\mathbf{x}_1 = (a, b)$, and sketch the vectors $\mathbf{u}_1$, $\mathbf{x}_1$, and $\mathbf{Q}_1\mathbf{x}_1$ in the plane.

(vi) Compute the reflection matrix $\mathbf{Q}_2 = \mathbf{I} - 2\mathbf{u}_2\mathbf{u}_2^T$ where $\mathbf{u}_2 = (0, \sqrt{2}/2, \sqrt{2}/2)$. Compute $\mathbf{Q}_2\mathbf{x}_2$, where $\mathbf{x}_2 = (1, 1, 1)$, and sketch the vectors $\mathbf{u}_2$, $\mathbf{x}_2$, and $\mathbf{Q}_2\mathbf{x}_2$ in $\mathbb{R}^3$.

Problem 3. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the (independent) column vectors of the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

whose column space is the orthogonal complement to the vector $(1, 1, 1, 1)$. Use the Gram-Schmidt process to construct an orthonormal basis $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$. Then write $\mathbf{M} = \mathbf{QR}$, where $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is upper-triangular.