

Topics: Vector spaces and subspaces

Do: Answer the following questions.

Problem 1.

Let X be a vector space over a field K . Let 0 be the zero element of K , and $\mathbf{0}$ the zero element of X . Using only the definitions of a group, vector space, and field, carefully prove the following.

- (i) $(-1)x = -x$ for all $x \in X$.
- (ii) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

Problem 2. Let X be a vector space.

- (i) Prove that for any collection $\{Y_\alpha \mid \alpha \in A\}$ of subspaces, $\bigcap_\alpha Y_\alpha$ is a subspace.
- (ii) Prove that for any subset $S \subseteq X$ of a vector space,

$$\text{Span}(S) = \bigcap_{S \subseteq Y_\alpha \leq X} Y_\alpha.$$

Here, the intersection is taken over all subspaces Y_α of X that contain S .

Problem 3. Y and Z be subspaces of a vector space X , with $Y + Z = X$. That is,

$$Y + Z := \{y + z \mid y \in Y, z \in Z\} = X.$$

Prove that the following are equivalent:

- (i) $Y \cap Z = \{0\}$.
- (ii) Every $x \in X$ can be written *uniquely* as $x = y + z$, for some $y \in Y$ and $z \in Z$.