

Topics: Spanning, linear independence, basis

Do: Answer the following questions. Throughout, assume that X is a finite-dimensional vector space over a field K .

Problem 1. Show that $B = \{x_1, \dots, x_n\}$ is a basis if and only if every $x \in X$ can be written *uniquely* using the vectors in B .

Problem 2. Let S be a linearly independent subset of X . Prove the *exchange property*: For any nonzero $x_0 \in \text{Span}(S)$, there exists $x_1 \in S$ such that $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\text{Span}(S)$.

Problem 3. Let $S = \{s_1, \dots, s_k\} \subset X$. Prove that the following are equivalent:

1. S is a *minimal* spanning set.
2. S is a *maximal* linear independent set.
3. S is a *basis*.