Topics: Basic category theory.

Do: Answer the following questions.

Problem 1. If $f\colon A \to B$ is a morphism in a category $\mathcal{C}$, then $g\colon B \to A$ is its *inverse* if $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$. Prove that if f has an inverse, then it is unique.

Problem 2. Let X be an object in a category $\mathcal{C}$. A *universal pair* (U, v) for Y with respect to a property consists of an object U and morphisms $v\colon U \to Y$, such that every other morphism $f\colon X \to Y$ with the same property factors through v uniquely. That is, there exists a unique morphism $g\colon X \to U$ such that $f = v \circ g$, i.e., it makes the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "g"'] & & Y \\ & U \arrow[ur, "v"'] & \end{tikzcd}$$

Prove that if Y has a universal pair (U, v) with respect to some property, then U is unique up to isomorphism.

Problem 3. Let A and B be objects in the category **Set**.

(i) Prove that the Cartesian product $A \times B$, together with the projection maps

$$\pi_A\colon A \times B \longrightarrow A, \qquad \pi_B\colon A \times B \longrightarrow B,$$

is a product of A and B in **Set**.

(ii) Prove that the disjoint union $A \sqcup B$, together with the inclusion maps

$$\iota_A\colon A \longrightarrow A \sqcup B, \qquad \iota_B\colon B \longrightarrow A \sqcup B,$$

is a coproduct of A and B in **Set**.

(iii) Explain why the ordinary union $A \cup B$ is not always a coproduct of A and B in **Set**.