Topics: The dual of a vector space.

Do: Answer the following questions.

Problem 1. Let X be a vector space over a field K and let X' be the set of linear functions from X to K, also known as the *dual space* of X.

(i) Let $x_1, \dots, x_n$ be a basis for X. Define $\ell_j \in X'$ by $\ell_j(x_i) = \delta_{ij}$. Show that $\ell_1, \dots, \ell_n$ is a basis for X'; it is called the *dual basis* of $x_1, \dots, x_n$.

(ii) Find the dual basis of $x_1 = (1, -1, 3)$, $x_2 = (0, 1, -1)$, and $x_3 = (0, 3, -2)$ in $X = \mathbb{R}^3$.

(iii) Express the scalar function $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, ℓ_1, ℓ_2, ℓ_3, from Part (b).

Problem 2. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ over $\mathbb{R}$, with degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define

$$\ell_j \colon \mathcal{P}_2 \longrightarrow \mathbb{R}, \qquad \ell_j(p) = p(\xi_j) \quad \text{for} \quad j = 1, 2, 3.$$

(i) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}_2'$.

(ii) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}_2'$.