Topics: The transpose of a linear map.

Do: Answer the following questions.

Problem 1. Recall that the *transpose* of a linear map $T\colon X \to U$ is the linear map $T'\colon U' \to X'$ between the dual spaces such that $(\ell, Tx) = (T'\ell, x)$ for all $x \in X$ and $\ell \in U'$. In class, we showed that $(ST)' = T'S'$. Prove the following.

(i) $T'' = T$.

(ii) $(S+T)' = S' + T'$, for $S, T\colon X \to U$.

(iii) $I' = I$, where $I\colon X \to X$ is the identity map.

(iv) $(T^{-1})' = (T')^{-1}$.

Problem 2. Let $T\colon X \to U$ be a linear map with range R_T and nullspace N_T. Recall that the *annihilator* of a subspace $Y \leq X$ is

$$Y^{\perp} = \{\ell \in X'\colon (\ell, y) = 0, \forall y \in Y\}.$$

In class, we showed that $R_T^{\perp} = N_{T'}$. As an immediate corollary, applying this statement to $T'\colon U' \to X'$ gives the analogous result $R_{T'}^{\perp} = N_T$. Give an alternate direct proof of this fact, akin to what we did in class for $R_T^{\perp} = N_{T'}$.