Topics: The matrix of a linear map; change-of-basis matrices.

Do: Answer the following questions.

Problem 1. Let $\mathcal{P}_n$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than n.

(i) Show that the map $T\colon \mathcal{P}_3 \to \mathcal{P}_4$ given by

$$T(p(x)) = 6\int_1^x p(t)\,dt$$

is linear. Determine whether it is 1–1 or onto.

(ii) Let $\mathcal{B}_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $\mathcal{B}_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

Problem 2. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show how to construct bases $\mathcal{B}_X$ for X and $\mathcal{B}_U$ for U such that the matrix of T in block form is

$$M = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix, and the other blocks are either empty or contain all zeros. Prove/justify all of your claims.

Problem 3. Let X be a vector space with basis x_1, x_2, x_3, and consider the linear map $T : X \to X$ with matrix representation $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to this basis. What is the matrix representation of T with respect to the basis $x_1 - x_2$, $x_2 - x_3$, $x_1 + x_3$.