

Topics: Eigenvalues and eigenvectors.

Do: Answer the following questions.

Problem 1. Find the eigenvalues and eigenvectors for the following matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

Problem 2. The *characteristic polynomial* of $\mathbf{A}$ is $\chi_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$. Suppose this factors as

$$\chi_{\mathbf{A}}(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

- (i) Plug in $t = 0$ and find a formula for $\det \mathbf{A}$ in terms of the eigenvalues of $\mathbf{A}$.
- (ii) The *trace* of $\mathbf{A}$, denoted $\text{tr } \mathbf{A}$, is the sum of the diagonal entries, which is also equal to the sum of the eigenvalues. If $\mathbf{A}$ is 2×2 , then

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(t\mathbf{I} - \mathbf{A}) = t^2 - (a + d)t + (ad - bc).$$

Write a formula for the characteristic polynomial of a 2×2 matrix in terms of $\det \mathbf{A}$ and $\text{tr } \mathbf{A}$.

- (iii) Suppose $\mathbf{A}$ is an $n \times n$ matrix with characteristic polynomial $\chi_{\mathbf{A}}(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1t + c_0$. Describe $\det \mathbf{A}$ and $\text{tr } \mathbf{A}$ in terms of the c_i 's.
- (iv) Explain why $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ is impossible for $n \times n$ matrices.

Problem 3. Suppose $\mathbf{A}$ is a 3×3 matrix with eigenvalues 0, 3, and 5, with respective eigenvectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

- (i) Give a basis for the nullspace and a basis for the column space.
- (ii) Find a particular solution to $\mathbf{Ax} = \mathbf{v} + \mathbf{w}$. Then, find all solutions.
- (iii) Explain why $\mathbf{Ax} = \mathbf{u}$ has no solution.