

Topics: Eigenvectors and generalized eigenvectors.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

The *characteristic polynomial* of an $n \times n$ matrix A is $p_A(t) := \det(tI - A)$. By the *Cayley Hamilton theorem*, $p_A(A) = 0$. The *minimal polynomial* is the smallest-degree monic polynomial $m_A(t)$ for which $m_A(A) = 0$, and it must divide $p_A(t)$.

Problem 1. Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

- (i) Find the characteristic and minimal polynomials of each.
- (ii) Diagonalize each matrix into $\mathbf{QDQ}^T$, where $\mathbf{Q}$ is a (real-valued) *orthogonal* matrix.
- (iii) Find *all* orthogonal matrices that diagonalize $\mathbf{A}$. How many will diagonalize $\mathbf{B}$?

Problem 2. Do the following for the matrix $\mathbf{A}$ from the previous worksheet, and then repeat it for $\mathbf{B}$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

- (i) For each eigenvalue λ , compute $\dim N_{(\mathbf{A} - \lambda \mathbf{I})^j}$ for $j = 1, 2, 3, \dots$
- (ii) Find the characteristic and minimal polynomials, and all (genuine) eigenvectors.
- (iii) Find a basis $\mathcal{B}$ of $\mathbb{C}^4$ consisting of generalized eigenvectors, so that the matrix with respect to this basis is $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where $\mathbf{J}$ is a *Jordan matrix*. This means that $\mathbf{J}$ is block-diagonal formed from *Jordan blocks* $\mathbf{J}_\lambda$; see above.

Problem 3. If $A: X \rightarrow X$ is a linear map, then a subspace $Y \subseteq X$ is *A-invariant* if $A(Y) \subseteq Y$. Show that for any scalar $\lambda \in K$, not necessarily an eigenvalue, the subspace Y is *A-invariant* if and only if it is $(A - \lambda I)$ -invariant.