Topics: Eigenspaces and Jordan canonical form.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

Problem 1. Consider the matrices $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(i) Decompose $\mathbb{R}^3$ into a direct sum of eigenspaces of each matrix.

(ii) Further decompose the 2-dimensional A-eigenspace as a direct sum of two 1-dimensonal B-eigenspaces, and vice-versa.

(iii) Write $\mathbb{R}^3$ as a direct sum of three 1-dimensional subspaces that are common eigenspaces of A and B, two different ways.

(iv) For each of your answers to Part (c), find a matrix P so that $P^{-1}AP = D_A$ and $P^{-1}BP = D_B$, where D_A and D_B are diagonal.

Problem 2. Let $\mathbf{A}$ be a 7×7 matrix over $\mathbb{C}$ with minimal polynomial $m(t) = (t-1)^3(t-2)^2$.

(i) List all possible Jordan canonical forms of A up to similarity.

(ii) For each matrix from Part (a), find the rank of $(\mathbf{A} - \mathbf{I})^k$ and $(\mathbf{A} - 2\mathbf{I})^k$, for $k \in \mathbb{N}$.

Problem 3.

Let A be an $n \times n$ matrix over $\mathbb{C}$. The matrix $\mathbf{A}$ is *nilpotent* if $\mathbf{A}^k = 0$ for some $k \in \mathbb{N}$, and $\mathbf{A}$ is *idempotent* if $\mathbf{A}^2 = \mathbf{A}$.

(i) Prove that if $\mathbf{A}^k = \mathbf{A}$ for some integer $k > 1$, then $\mathbf{A}$ is diagonalizable.

(ii) Prove that idempotent matrices are similar if and only if they have the same trace.

(iii) Prove that if $\mathbf{A}$ is nilpotent, then $A^n = 0$.

(iv) Prove that if $\mathbf{A}$ is nilpotent, then there is some $r \in \mathbb{N}$ and positive integers $k_1 \geq \cdots \geq k_r$ with $k_1 + \cdots + k_r = n$ that determine A up to similarity.

(v) Suppose $\mathbf{A}$ and $\mathbf{B}$ are 6×6 nilpotent matrices with the same minimal polynomial and $\dim N_{\mathbf{A}} = \dim N_{\mathbf{B}}$. Prove that $\mathbf{A}$ and $\mathbf{B}$ are similar. Show by example that this can fail for 7×7 matrices.