

Topics: Singular value decomposition, pseudo-inverses.

Do: Answer the following questions. Assume that all matrices are over the field $K = \mathbb{C}$.

Problem 1. Consider the following matrices

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}, \quad A A^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}.$$

- (i) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors v_1, v_2 of $A^T A$.
- (ii) For the $\sigma_i \neq 0$, compute $u_i = A v_i / \sigma_i$ and verify that indeed $\|u_i\| = 1$. Find the other u_i by computing the other unit eigenvector of $A A^T$.
- (iii) Construct the singular value decomposition (SVD), $A = U \Sigma V^T$.
- (iv) Write down orthonormal bases for each the “four fundamental subspaces”: the column space R_A , the nullspace N_A , the row space R_{A^T} , and the left nullspace N_{A^T} .
- (v) Describe *all* matrices that have the same four fundamental subspaces.
- (vi) Find a left, right, and pseudoinverse of A , or explain why it doesn't exist.

Problem 2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

- (i) Construct the singular value decomposition of A .
- (ii) Write down orthonormal bases for each the “four fundamental subspaces”: the row space R_A , the nullspace N_A , the column space R_{A^T} , and the left nullspace N_{A^T} .
- (iii) Find a left inverse, right inverse, and pseudoinverse of A , or explain why it doesn't exist.

Problem 3. Let $f: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear map defined by $f(x) = Mx$ for $x \in \mathbb{R}^4$ where $M = ABC$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

1. Define the adjoint map $f^*: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ (under the standard Euclidean inner product) and express it in terms of M .
2. Find a singular value decomposition (SVD) of M . (*Hint:* Observe that $A^T A$ and $C^T C$ are diagonal.)
3. Find all $x \in \mathbb{R}^4$ with $\|x\| = 1$ so that $\|Mx\|$ is maximized.
4. Describe the eigenvalues and eigenvectors of $M^T M$.
5. Find the least square solution for $Mx = b$ with $\|x\|_2$ minimal where $b = (1, 1, 1)^T$. (*Hint:* Use the pseudo-inverse of M .)