Group actions

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Summer Bridge Course

Conjugation preserves structure

Think back to linear algebra. Matrices A and B are similar (=conjugate) if $A = PBP^{-1}$.

Conjugate matrices have the same eigenvalues, eigenvectors, and determinant.

In fact, they represent the same linear map, but under a change of basis.

$$
\begin{tikzcd}
\textit{new basis} & X \arrow[r, "D"] \arrow[d, "P"'] & X \arrow[d, "P"] \\
\textit{old basis} & X \arrow[r, "A"] & X
\end{tikzcd}
$$

Central theme in mathematics

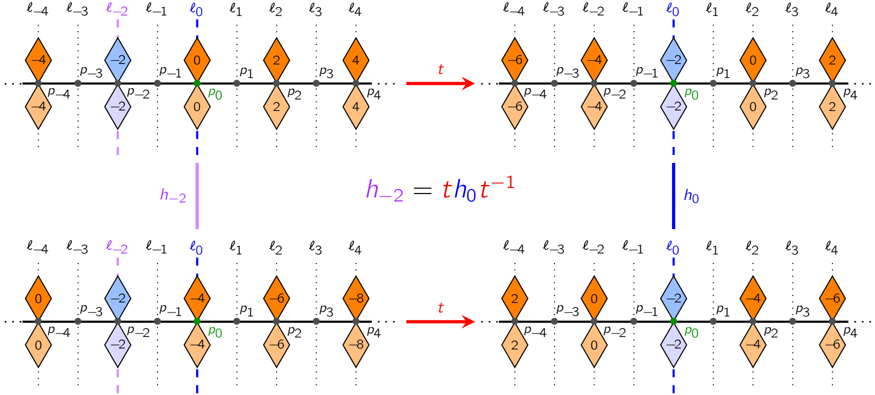
Two things that are conjugate have the same structure.

Conjugation preserves structure

To understand what we mean by conjugation preserves structure, let's revisit frieze groups.

Let $h = h_0$ denote the reflection across the central axis, ℓ_0.

Suppose we want to reflect across a different axis, say ℓ_{-2}.



It should be clear that all reflections (resp., rotations) of the "same parity" are conjugate.

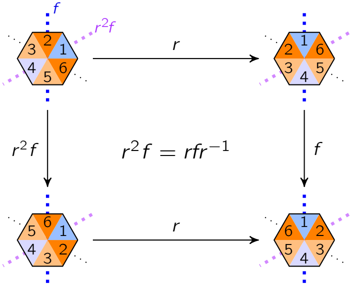
Conjugacy classes in D_n

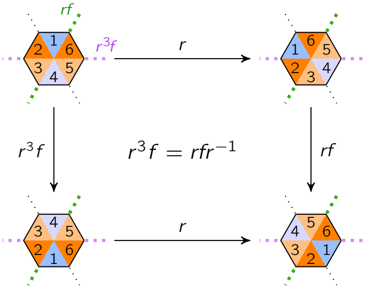
The dihedral group D_n is a "finite version" of the previous frieze group.

When n is even, there are two "*types of reflections*" of an n-gon:

1. $r^{2k}f$ is across an axis that bisects two sides
2. $r^{2k+1}f$ is across an axis that goes through two corners.

Here is a visual reason why each of these two types form a conjugacy class in D_n.





What do you think the conjugacy classes of a reflection is in D_n when n is odd?

Next, let's verify the conjugacy classes algebraically.

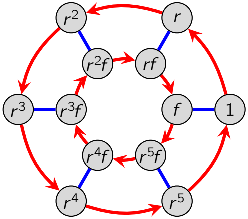
Conjugacy classes in D_6

The center is $Z(D_6) = \{1, r^3\}$; these are the *only* elements in size-1 conjugacy classes.

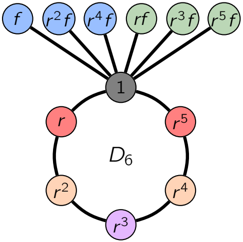
The only two elements of order 6 are r and r^5, so $\mathrm{cl}_{D_6}(r) = \{r, r^5\}$.

The only two elements of order 3 are r^2 and r^4, so $\mathrm{cl}_{D_6}(r^2) = \{r^2, r^4\}$.

Two reflections $r^i f$ and $r^k f$ are conjugate iff i and k have the same parity.

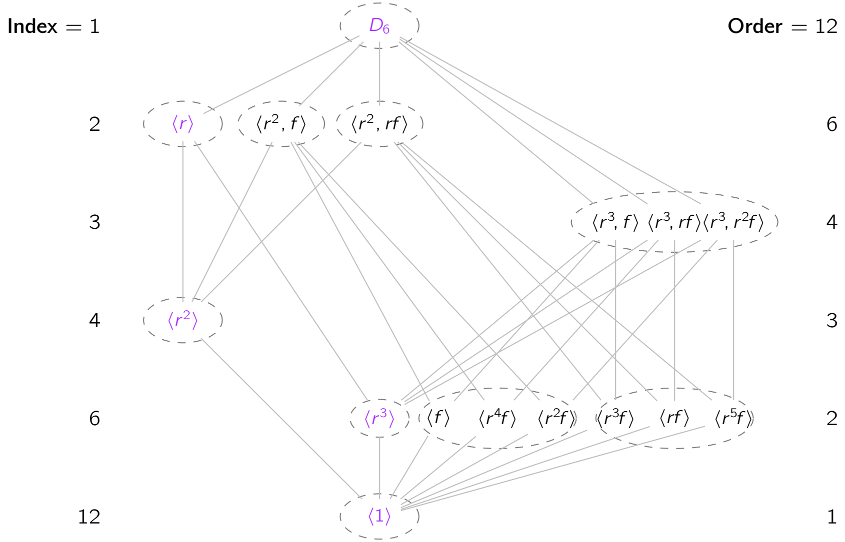


1	r	r^2	f	r^2f	r^4f
r^3	r^5	r^4	rf	r^3f	r^5f

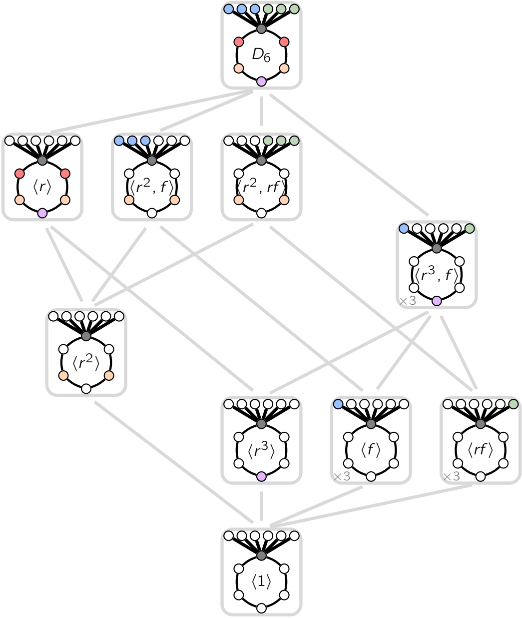


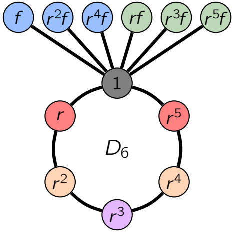
The subgroup lattice of D_6

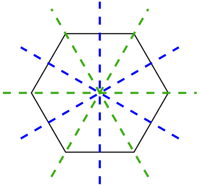
We can now deduce the conjugacy classes of the subgroups of D_6.



The subgroup diagram of D_6

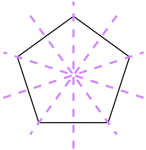


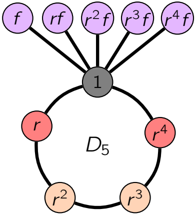


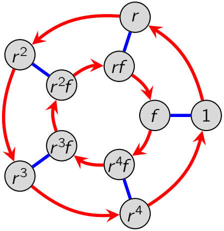


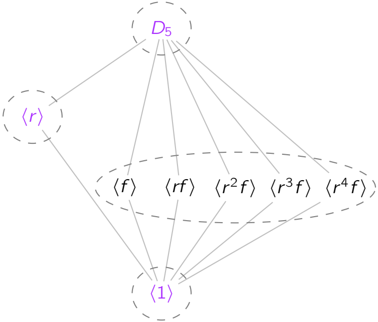
Conjugacy classes in D_5

Since $n = 5$ is odd, all reflections in D_5 are conjugate.









1	rf	r^3f	r	r^4
f	r^2f	r^4f	r^2	r^3

Cycle type and conjugacy in the symmetric group

This is best seen by example. There are five cycle types in S_4:

example element	e	(12)	(234)	(1234)	(12)(34)
parity	even	odd	even	odd	even
# elts	1	6	8	6	3

Definition

Two elements in S_n have the same cycle type if when written as a product of disjoint cycles, there are the same number of length-k cycles for each k.

Theorem

Two elements $g, h \in S_n$ are conjugate if and only if they have the same cycle type.

For example, permutations in S_5 fall into seven cycle types (conjugacy classes):

$$\mathrm{cl}(e), \quad \mathrm{cl}((12)), \quad \mathrm{cl}((123)), \quad \mathrm{cl}((1234)), \quad \mathrm{cl}((12345)), \quad \mathrm{cl}((12)(34)), \quad \mathrm{cl}((12)(345)).$$

Big idea

Conjugate permutations have the same structure: they are *the same up to renumbering*.

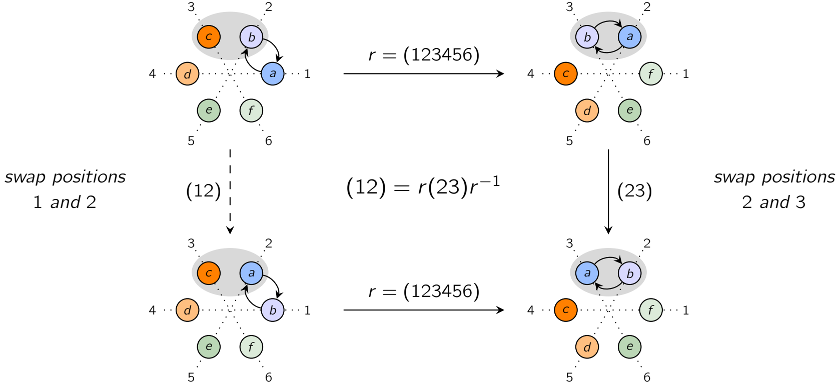
Conjugation preserves structure in the symmetric group

The symmetric group $G = S_6$ is generated by a transposition $(i\ i+1)$ and an n-cycle.

Consider the permutations of seating assignments around a circular table achievable by

- (23): "*people in chairs* 2 *and* 3 *may swap seats*"
- (123456): "*people may cyclically rotate seats counterclockwise*"

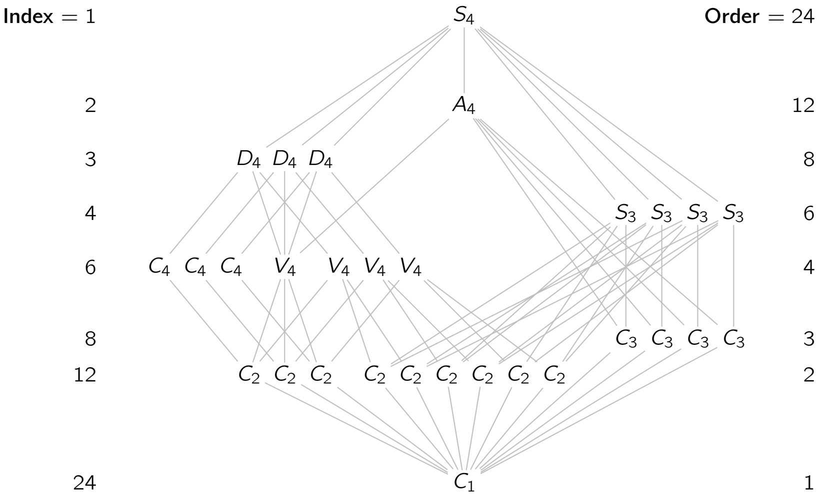
Here's how to get people in chairs 1 and 2 to swap seats:



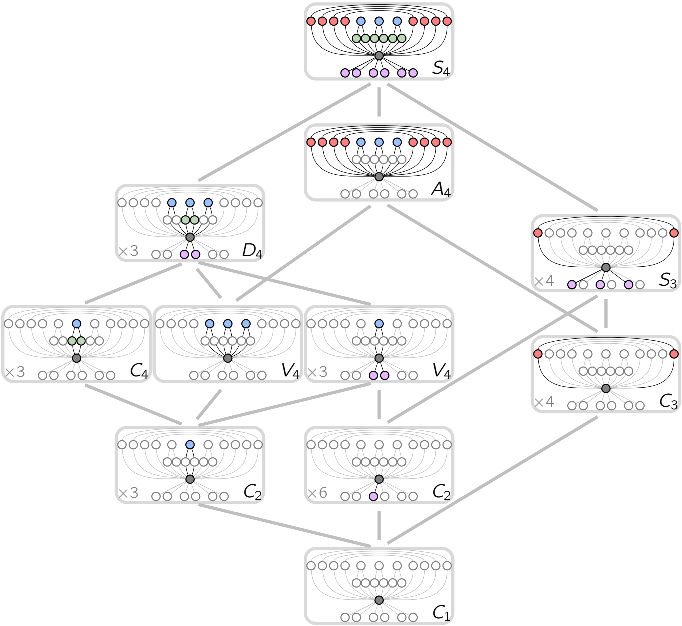
The subgroup lattice of S_4

Exercise

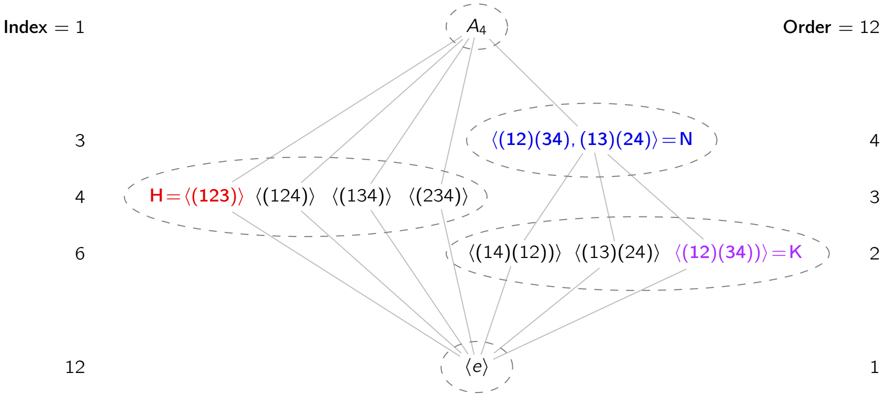
Partition the subgroup lattice of S_4 into conjugacy classes by inspection alone.



The subgroup diagram of S_4



The subgroup lattice of A_4



Going forward, we will consider the following three subgroups of A_4:

$$N = \big\langle (12)(34), (13)(24) \big\rangle = \big\{e,\ (12)(34),\ (13)(24),\ (14)(23)\big\} \cong V_4$$
$$H = \big\langle (123) \big\rangle = \big\{e,\ (123),\ (132)\big\} \cong C_3$$
$$K = \big\langle (12)(34) \big\rangle = \big\{e,\ (12)(34)\big\} \cong C_2.$$

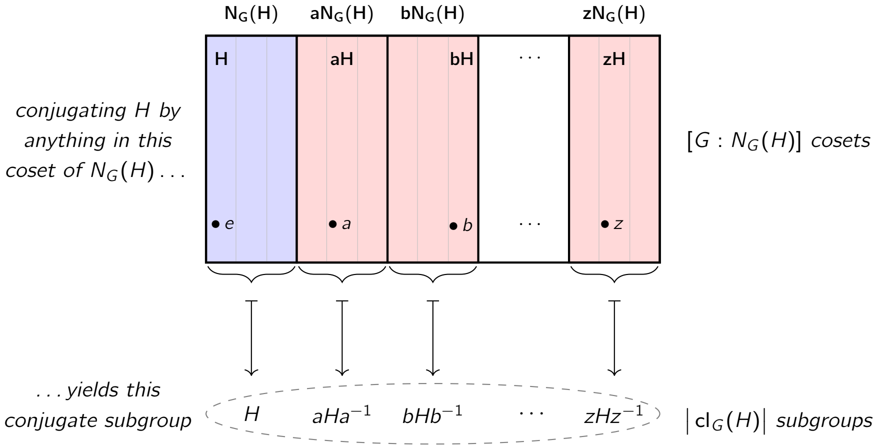
For each one, its normalizer lies between it and A_4 (inclusive) on the subgroup lattice.

The number of conjugate subgroups

Theorem

For any subgroup $H \leq G$, the **size of its conjugacy class** is the **index of its normalizer**:

$$\left|\mathrm{cl}_G(H)\right| = [G : N_G(H)].$$



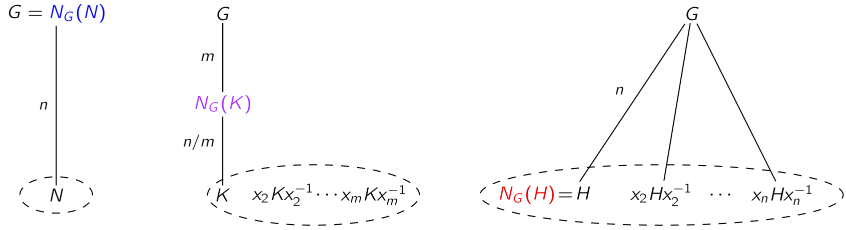
The number of conjugate subgroups

Theorem

Let $H \leq G$ with $[G:H] = n < \infty$. Then

$$\big|\mathrm{cl}_G(H)\big| = [G : N_G(H)] = \frac{\#\text{ elts } x \in G}{\#\text{ elts } x \in G \text{ for which } xH = Hx} = \frac{1}{\mathrm{Deg}_G^{\triangleleft}(H)}.$$

That is, H has exactly $[G : N_G(H)]$ conjugate subgroups.

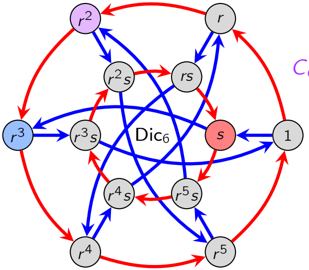


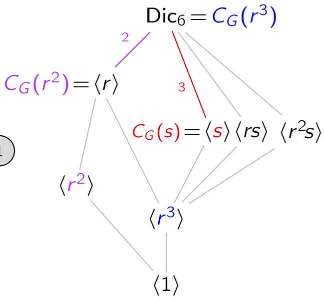
normal — $\big|\mathrm{cl}_G(N)\big| = 1$

moderately unnormal — $1 < \big|\mathrm{cl}_G(K)\big| < [G:K]$

fully unnormal — $\big|\mathrm{cl}_G(H)\big| = [G:H]$; as large as possible

An example: conjugacy classes and centralizers in Dic_6





rs	r^3s	r^5s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

conjugacy classes

r^2	r^5	r^2s	r^5s
r	r^4	rs	r^4s
1	r^3	s	r^3s

$[G : C_G(r^3)] = 1$
"central"

rs	r^3s	r^5s
s	r^2s	r^4s
r	r^3	r^5
1	r^2	r^4

$[G : C_G(r^2)] = 2$
"moderately uncentral"

r^2	r^2s	r^5	r^5s
r	rs	r^4	r^4s
1	s	r^3	r^3s

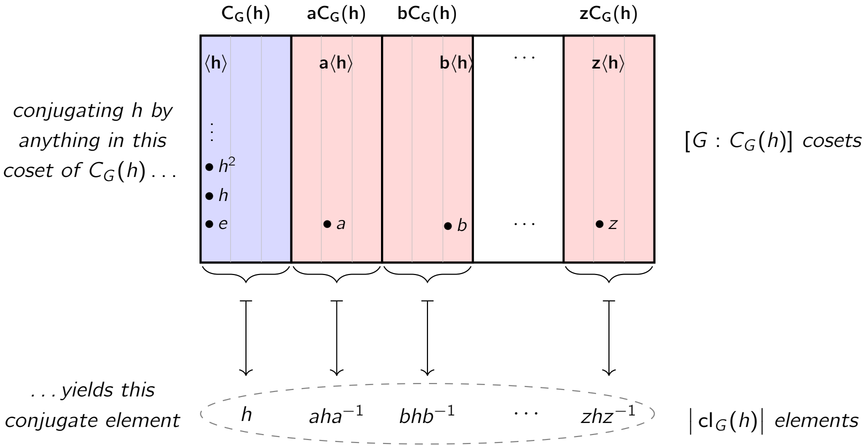
$[G : C_G(s)] = 3$
"fully unncentral"

The number of conjugate elements

Theorem

For any element $h \leq G$, the **size of its conjugacy class** is the **index of its centralizer**:

$$\left|\mathrm{cl}_G(h)\right| = [G : C_G(h)].$$



The number of conjugate elements

The following result is analogous to an earlier one on the degree of normality and $|\operatorname{cl}_G(H)|$.

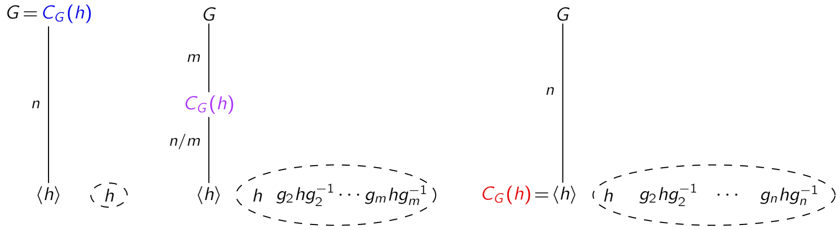
Theorem

Let $h \in G$ with $[G : \langle h \rangle] = n < \infty$. Then

$$\left|\operatorname{cl}_G(h)\right| = [G : C_G(h)] = \frac{\#\text{ elts } x \in G}{\#\text{ elts } x \in G \text{ for which } xh = hx} = \frac{1}{\operatorname{Deg}_G^C(h)}.$$

That is, there are exactly $[G : C_G(h)]$ elements conjugate to h.

Both of these are special cases of the **orbit-stabilizer theorem**, about group actions.



central

$\left|\operatorname{cl}_G(h)\right| = 1$

moderately uncentral

$1 < \left|\operatorname{cl}_G(h)\right| < [G : \langle h \rangle]$

fully uncentral

$\left|\operatorname{cl}_G(h)\right| = [G : \langle h \rangle]$; as large as possible

Conjugacy class size

Theorem (number of conjugate subgroups)

The conjugacy class of $H \leq G$ contains exactly $[G : N_G(H)]$ subgroups.

Proof (roadmap)

Construct a bijection between left cosets of $N_G(H)$ and conjugate subgroups of H:

"$xHx^{-1} = yHy^{-1}$ *iff x and y are in the same left coset of* $N_G(H)$."

Define $\phi \colon \{\text{left cosets of } N_G(H)\} \longrightarrow \{\text{conjugates of } H\}$, $\qquad \phi \colon xN_G(H) \longmapsto xHx^{-1}$.

Theorem (number of conjugate elements)

The conjugacy class of $h \in G$ contains exactly $[G : C_G(h)]$ elements.

Proof (roadmap)

Construct a bijection between left cosets of $C_G(h)$, and elements in $\mathrm{cl}_G(h)$:

"$xhx^{-1} = yhy^{-1}$ *iff* x and y are in the same left coset of $C_G(h)$."

Define $\phi \colon \{\text{left cosets of } C_G(h)\} \longrightarrow \{\text{conjugates of } h\}$, $\qquad \phi \colon xC_G(h) \longmapsto xhx^{-1}$.

Conjugacy class summary: subgroups vs. elements

The relationship between conjugacy classes and cosets of a certain subgroup are analogous.

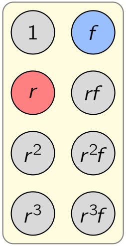
	conjugacy of subgroups	conjugacy of elements
objects	$H \leq G$	$h \in G$
conjugacy class	$\mathrm{cl}_G(H)$	$\mathrm{cl}_G(h)$
they partition	the subgroups of G	G
-izer subgroup	Normalizer $N_G(H)$	Centralizer $C_G(h)$
$\lvert$conj. class$\rvert$	$[G : N_G(H)]$	$[G : C_G(h)]$
"best case" (size-1 class)	$N_G(H)=G$ (iff $H \trianglelefteq G$)	$C_G(h)=G$ (iff $h \in Z(G)$)
"worst case" (max'l class)	$N_G(H)=H$; "*fully unnormal*"	$C_G(h)=\langle h\rangle$; "*fully uncentral*"

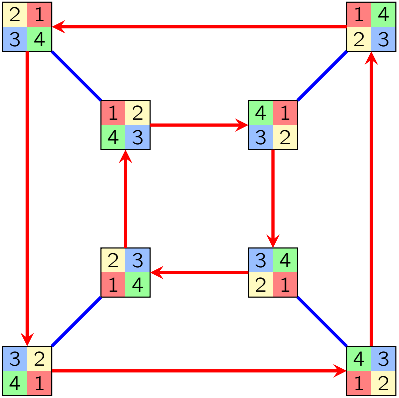
Both are special cases of the orbit-stabilizer theorem, from Chapter 5 (group actions).

Group actions

Here is the action graph of the group $D_4 = \langle r, f \rangle$ acting on the set of 8 configurations of the square:

"Group switchboard"





There is *bijection* between configurations and actions (group elements).

This need not happen in general.

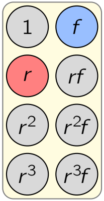
Action graphs

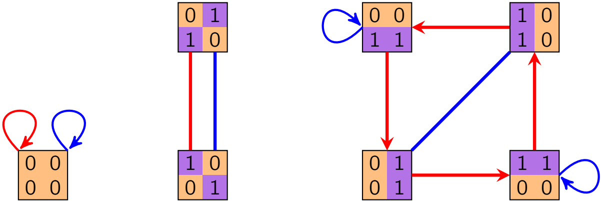
Suppose we have a size-7 set consisting of the following "binary squares."

$$S = \left\{ \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix} \right\}$$

The group $D_4 = \langle r, f \rangle$ "acts on S" as follows:

"Group switchboard"





The "group switchboard" analogy

Suppose we have a "switchboard" for G, with every element $g \in G$ having a "button."

If $a \in G$, then pressing the a-button rearranges the objects in S—it is a permutation of S; call it $\phi(a)$.

If $b \in G$, then pressing the b-button also rearranges the objects in S. Call this permutation $\phi(b)$.

The element $ab \in G$ also has a button. We require that pressing the ab-button does the same as pressing the a-button, followed by the b-button. That is,

$$\phi(ab) = \phi(a)\phi(b)\,, \qquad \text{for all } a, b \in G\,.$$

Let $\mathbf{Perm}(S)$ be the group of permutations of S. Thus, if $|S| = n$, then $\mathbf{Perm}(S) \cong S_n$. (We typically think of S_n as the permutations of $\{1, 2, \ldots, n\}$.)

Definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \to \mathbf{Perm}(S)$.

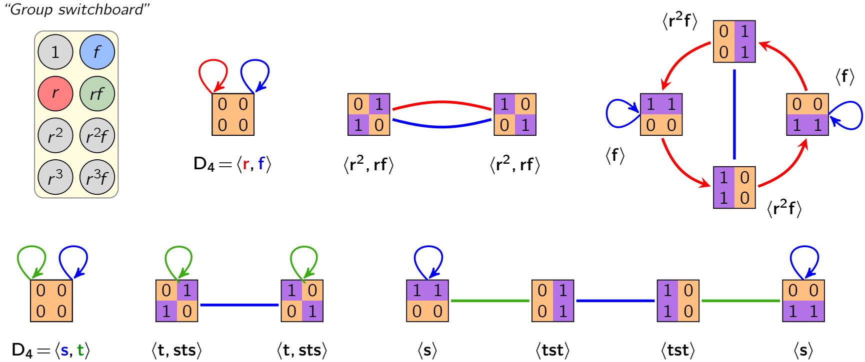
Action graphs vs. G-sets

Definition

A set S with an action by G is called a (right) G-set.

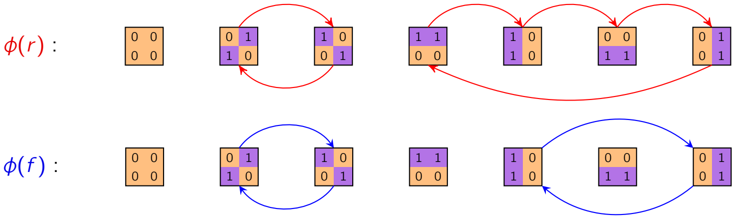
Big ideas

- An action $\phi\colon G \to \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

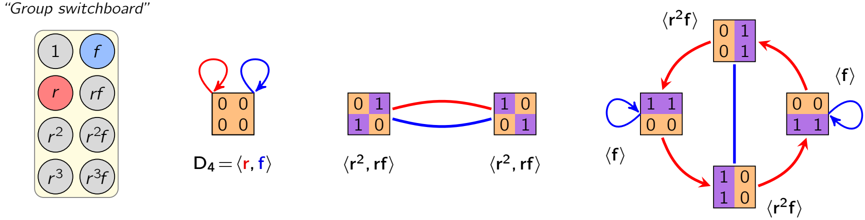


The “group switchboard” analogy

In our binary square example, pressing the r-button and f-button permutes S as follows:



Observe how these permutations are encoded in the action graph. (Next to each $s \in S$ is the subgroup that fixes it.)

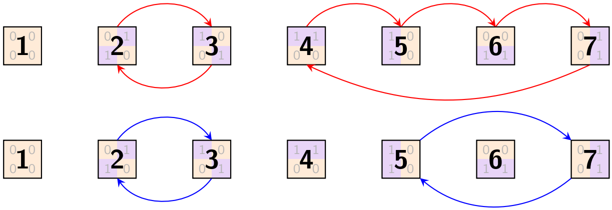


The "group switchboard" analogy

This action is an embedding $\phi \colon D_4 \hookrightarrow \mathrm{Perm}(S) \cong S_7$.

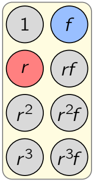
$\phi(r)$:

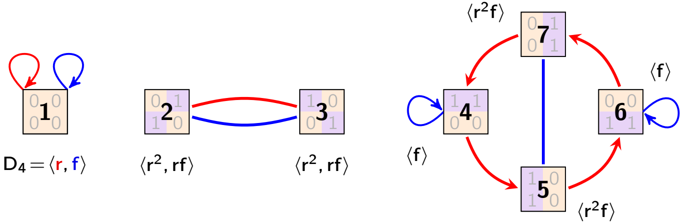
$\phi(f)$:



Notice that $\mathrm{Im}(\phi) = \big\langle (23)(4567), (23)(57) \big\rangle \cong D_4 \leq S_7$.

"Group switchboard"





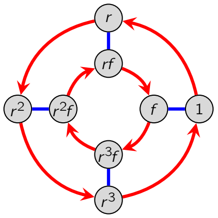
G-sets generalize groups. Action graphs generalize Cayley graphs

The group $G = D_4 = \langle r, f \rangle$ can act on itself ($S = D_4$), or on its subgroups,

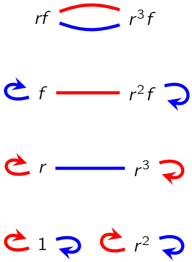
$$S = \big\{ D_4, \langle r\rangle, \langle r^2, f\rangle, \langle r^2, rf\rangle, \langle f\rangle, \langle rf\rangle, \langle r^2 f\rangle, \langle r^3 f\rangle, \langle r^2\rangle, \langle 1\rangle \big\}.$$

There are several ways to define the result of *"pressing the g-button on our switchboard"*.

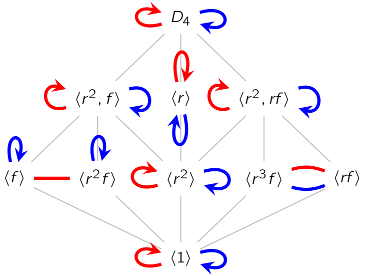
We say that: "*G acts on...*"



"... itself by right-multiplication"



"... itself by conjugation"



"... its subgroups by conjugation"

Big idea

Every Cayley graph is the action graph of a particular group action.

Left actions vs. right actions (an annoyance we can deal with)

As we've defined group actions, "*pressing the a-button followed by the b-button should be the same as pressing the ab-button*."

However, sometimes it appears like it's the same as "*pressing the ba-button*."

This is best seen by an example. Suppose our action is conjugation:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

We'll call aHa^{-1} the left conjugate of H by a, and $a^{-1}Ha$ the right conjugate.

Some books forgo our "ϕ-notation" and use the following notation to distinguish left vs. right group actions:

$$g.(h.s) = (gh).s, \qquad (s.g).h = s.(gh).$$

We'll usually keep the ϕ, and write $\phi(g)\phi(h)s = \phi(gh)s$ and $s.\phi(g)\phi(h) = s.\phi(gh)$. As with groups, the "dot" will be optional.

Five features of every group action

Every group action has **five fundamental features** that we will always try to understand.

There are several ways to classify them. For example:

- three are subsets of S
- two are subgroups of G.

Another way to distinguish them is by local vs. global:

- three are features of individual group or set elements (we'll write in *lowercase*)
- two are features of the homomorphism ϕ. (we'll write in *Uppercase*)

We will see parallels within and between these classes.

For example, two "local" features will be "dual" to each other, as will the global features.

Our global features can be expressed as intersections of our local features, either ranging over all $s \in S$, or over all $g \in G$.

We'll start by exploring the three local features.

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Two local features: orbits and stabilizers

Suppose G acts on a set S, and pick some $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

(ii) The stabilizer of s in G is

$$\mathrm{stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

In terms of the action graph

(i) The orbit of $s \in S$ is the connected component containing s.

(ii) The stabilizer of $s \in S$ are the group elements whose paths start and end at s; "loops."

The third local feature: fixators

Our first two local features were specific to a certain element $s \in S$.

Our last local feature is defined for each group element $g \in G$. A natural question to ask is:

(iii) What *states* (in S) does g fix?

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(iii) The fixator of $g \in G$ are the elements $s \in S$ fixed by g:

$$\mathrm{fix}(g) = \big\{ s \in S \mid s.\phi(g) = s \big\}.$$

In terms of the action graph

(iii) The fixator of $g \in G$ are the nodes from which the g-paths are loops.

In terms of the "group switchboard analogy"

(i) The orbit of $s \in S$ are the elements in S that can be reached by pressing some combination of buttons.

(ii) The stabilizer of $s \in S$ consists of the buttons that have no effect on s.

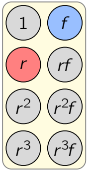
(iii) The fixator of $g \in G$ are the elements in S that don't move when we press the g-button.

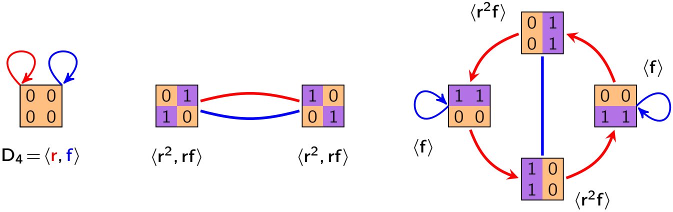
Three local features: orbits, stabilizers, and fixators

The orbits of our running example are the 3 connected components.

Each node is labeled by its stabilizer.

"Group switchboard"





The fixators are $\text{fix}(1) = S$, and

$$\text{fix}(r) = \text{fix}(r^3) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right\} \qquad \text{fix}(r^2) = \text{fix}(rf) = \text{fix}(r^3 f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\}$$

$$\text{fix}(f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \right\} \qquad \text{fix}(r^2 f) = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right\}$$

Local duality: stabilizers vs. fixators

Consider the following table, where a checkmark at (g, s) means "*g fixes s*."

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the fixators can be read off the rows: *set elements fixed by $g \in G$*.

The stabilizer subgroup

As we've seen, elements in the same orbit can have different stabilizers.

Proposition (HW exercise)

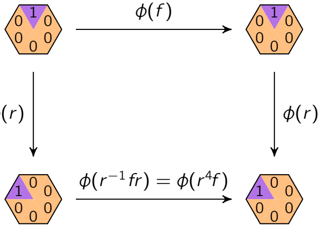
Set elements in the same orbit have conjugate stabilizers:

$$\operatorname{stab}(s.\phi(g)) = g^{-1}\operatorname{stab}(s)g, \quad \text{for all } g \in G \text{ and } s \in S.$$

In other words, if x stabilizes s, then $g^{-1}xg$ stabilizes $s.\phi(g)$.

Here are several ways to visualize what this means and why.

$$\begin{tikzcd}
s \arrow[r, mapsto, "\phi(x)"] \arrow[d, mapsto, "\phi(g)"'] & s \arrow[d, mapsto, "\phi(g)"] \\
s' \arrow[r, mapsto, "\phi(g^{-1}xg)"] & s'
\end{tikzcd}$$

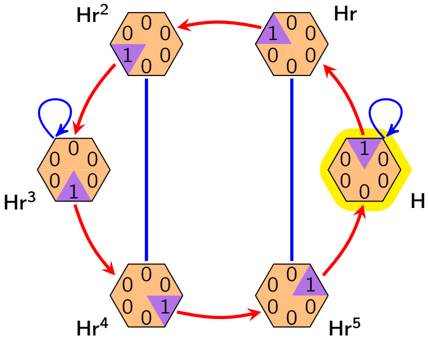


In other words, if x is a loop from s, and $s \xrightarrow{g} s'$, then $g^{-1}xg$ is a loop from s'.

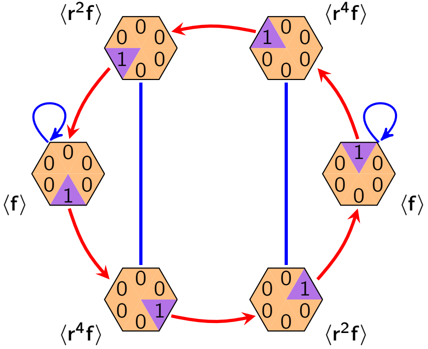
The stabilizer subgroup

Here is another example of an action (or G-set), this time of $G = D_6$.

Let s be the highlighted hexagon, and $H = \mathsf{stab}(s)$.



labeled by destinations



labeled by stabilizers

Two global features: fixed points and the kernel

Our last two features are properties of the action ϕ, rather than of specific elements.

One definition is new, and the othher is an familiar concept in this new setting.

Definition

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$.

(iv) The kernel of the action is the set

$$\mathsf{Ker}(\phi) = \big\{k \in G \mid \phi(k) = e\big\} = \big\{k \in G \mid s.\phi(k) = s \text{ for all } s \in S\big\}.$$

(v) The fixed points of the action, denoted $\mathsf{Fix}(\phi)$, are the orbits of size 1:

$$\mathsf{Fix}(\phi) = \big\{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\big\}.$$

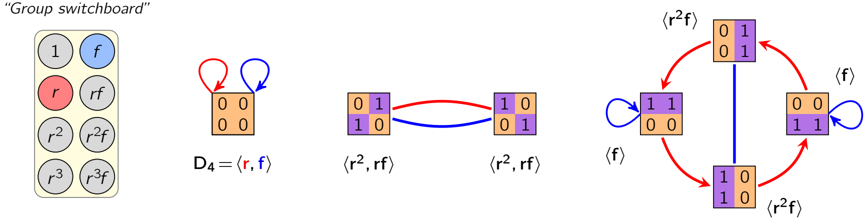
Proposition (global duality: fixed points vs. kernel)

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$\mathsf{Ker}(\phi) = \bigcap_{s \in S} \mathsf{stab}(s), \qquad \text{and} \qquad \mathsf{Fix}(\phi) = \bigcap_{g \in G} \mathsf{fix}(g).$$

Let's also write $\mathsf{Orb}(\phi)$ for the set of orbits of ϕ.

Two global features: fixed points and the kernel



In terms of the action graph

(iv) The kernel of ϕ are the paths that are "loops from every $s \in S$."

(v) The fixed points of ϕ are the size-1 connected components.

In terms of the group switchboard analogy

(iv) The kernel of ϕ are the "broken buttons"; those $g \in G$ that have no effect on any s.

(v) The fixed points of ϕ are those $s \in S$ that are not moved by pressing any button.

Global duality: fixed points vs. kernel

Consider the following table, where a checkmark at (g, s) means g fixes s.

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
r	$\checkmark$						
r^2	$\checkmark$	$\checkmark$	$\checkmark$				
r^3	$\checkmark$						
f	$\checkmark$			$\checkmark$		$\checkmark$	
rf	$\checkmark$	$\checkmark$	$\checkmark$				
r^2f	$\checkmark$				$\checkmark$		$\checkmark$
r^3f	$\checkmark$	$\checkmark$	$\checkmark$				

- the fixed points consist of columns with all checkmarks: *set elts fixed by everything*
- the kernel consists of the rows with all checkmarks: *group elements that fix everything*.

Two theorems on orbits, and their consequences

Our binary square example gives us some key intutition about group actions.

Qualitative observations

- elements in larger orbits tend to have smaller stabilizers, and vice-versa
- actions whose fixed point tables have more "checkmarks" tend to have more orbits.

Both of these qualitative observations can be formalized into quantitative theorems.

Theorems

1. **Orbit-stabilizer theorem**: the size of an orbit is the index of the stabilizer.
2. **Orbit-counting theorem**: the number of orbits is the average number of things fixed by a group element.

If we set up our group actions correctly, the orbit-stabilizer theorem will imply:

- The size of the conjugacy class $\mathrm{cl}_G(H)$ is the index of the normalizer of $H \leq G$
- The size of the conjugacy class $\mathrm{cl}_G(x)$ is the index of the centralizer of $x \in G$

We can also determine the number of conjugacy classes from the orbit-counting theorem.

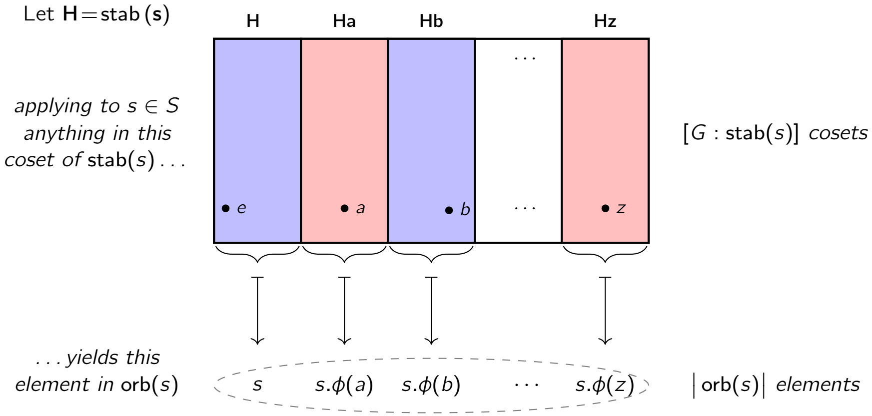
Our first theorem on orbits

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and $s \in S$, the size of the orbit containing s is

$$|\,\mathrm{orb}(s)| = [G : \mathrm{stab}(s)].$$

By Lagrange's theorem, this says that $|\,\mathrm{orb}(s)| \cdot |\,\mathrm{stab}(s)| = |G|$.



Our second theorem on orbits

Orbit-counting theorem

Let a finite group G act on a set S via $\phi \colon G \to \mathsf{Perm}(S)$. Then

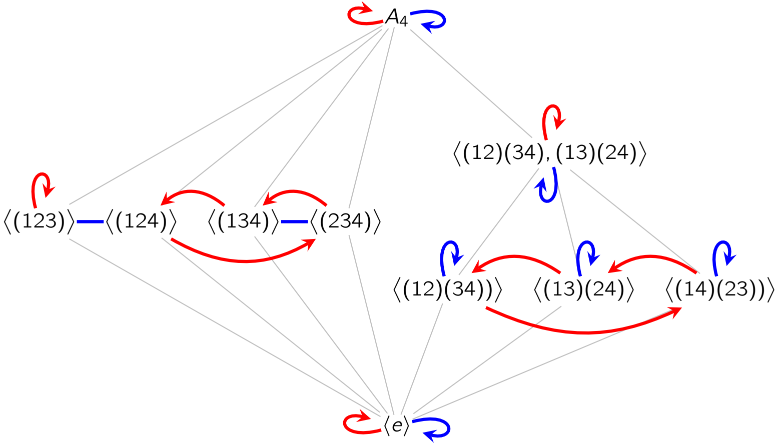
$$|\operatorname{Orb}(\phi)| = \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|.$$

This says that the "*average number of checkmarks per row*" is the number of orbits:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from earlier.

$$N = \big\langle (12)(34), (13)(24) \big\rangle, \qquad H = \big\langle (123) \big\rangle, \qquad K = \big\langle (12)(34) \big\rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e\rangle$	$\langle(123)\rangle$	$\langle(124)\rangle$	$\langle(134)\rangle$	$\langle(234)\rangle$	$\langle(12)(34)\rangle$	$\langle(13)(24)\rangle$	$\langle(14)(23)\rangle$	$\langle(12)(34),(13)(24)\rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
$(13)(24)$	✓					✓	✓	✓	✓	✓
$(14)(23)$	✓					✓	✓	✓	✓	✓

By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

Groups acting on cosets of H by multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = H\backslash G$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \;\text{ sends }\; Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\mathrm{stab}(Hx) = \big\{g \in G \mid Hxg = Hx\big\} = \big\{g \in G \mid Hxgx^{-1} = H\big\} = x^{-1}Hx.$$

- There doesn't seem to be a standard term for the fixator of g:

$$\mathrm{fix}(g) = \big\{Hx \mid Hxg = Hx\big\} = \big\{Hx \mid xgx^{-1} \in H\big\}.$$

- Assuming $H \neq G$, there are no fixed points of ϕ.

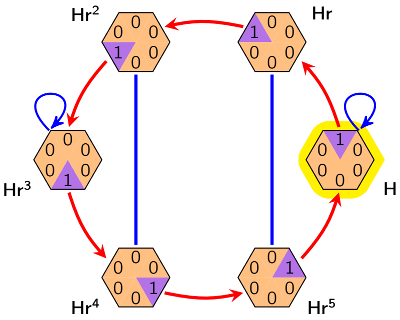
- The kernel of this action is the intersection of all conjugate subgroups of H:

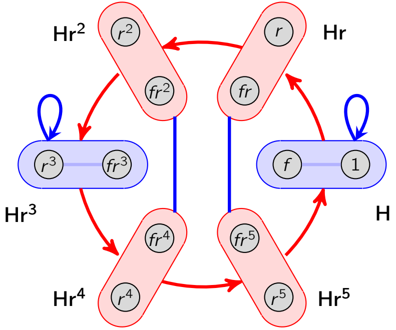
$$\mathrm{Ker}(\phi) = \bigcap_{x\in G} \mathrm{stab}(x) = \bigcap_{x\in G} x^{-1}Hx.$$

  Notice that $\langle 1\rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}(\phi) = H$ iff $H \trianglelefteq G$.

Groups acting on cosets of H by multiplication

Soon, we'll see that *every* transitive action is equivalent to G acting on cosets of a subgroup.





This is why it's helpful to have a notion of G-set isomorphism.

In other words, we can *always* quotient by a subgroup $H \leq G$ to get a G-set.

This G-set is a group if and only if H is normal.

Action equivalence

Let's recall the difference between left-conjugating and right conjugating:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

There's a better way to describe left actions than the faux-homomorphic $\phi(a)\phi(b) = \phi(ba)$.

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a^{-1}"] \arrow[rr, bend right, "\text{right-conj. by } (a^{-1}b^{-1})^{-1}"'] & aHa^{-1} \arrow[r, "\text{right-conjugate by } b^{-1}"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a^{-1})\phi(b^{-1}) = \phi(a^{-1}b^{-1}) = \phi((ba)^{-1})$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a"] \arrow[rr, bend right, "\text{right-conj. by } ab"'] & a^{-1}Ha \arrow[r, "\text{right-conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

Big idea

For every right action, there is an "equivalent" left-action where:

"pressing g-buttons, from L-to-R" $\Leftrightarrow$ "pressing g^{-1}-buttons, from R-to-L".

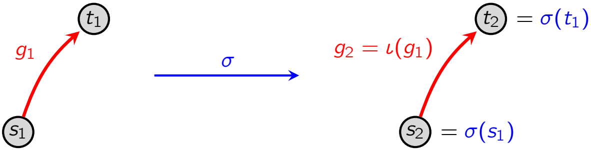
Action equivalence, informally

Action equivalence is more general. Consider two groups acting on sets, say via

$$\phi_1\colon G_1 \longrightarrow \mathsf{Perm}(S_1), \qquad \text{and} \qquad \phi_2\colon G_2 \longrightarrow \mathsf{Perm}(S_2).$$

If these are "equivalent", then we'll need

- a set bijection $\sigma\colon S_1 \longrightarrow S_2$
- a group isomorphism $\iota\colon G_1 \longrightarrow G_2$.



Informally, these actions are **equivalent** if:

1. pressing the g_1-button in the G_1-switchboard, followed by
2. applying $\sigma\colon S_1 \to S_2$ to get to the other graph

is the same as doing these steps in reverse order. That is,

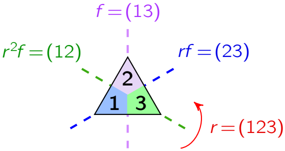
1. applying $\sigma\colon S_1 \to S_2$ to get to the other graph, then
2. pressing the $\iota(g_1)$-button on the G_2-switchboard.

A familiar example of equivalent actions

We've seen the groups:

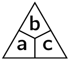
- D_3 act on a set X of six triangles,
- S_3 act on a set X' of six permuations of **123**.

These two actions are equivalent.



$$X' = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}$$

Set bijection

σ : 

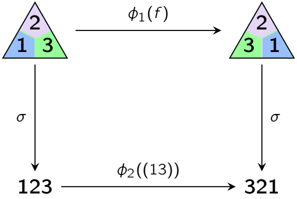
$\longmapsto$ **abc**

Isomorphism

$\iota \colon D_3 \longrightarrow S_3$

$\iota \colon r \longmapsto (123)$

$\iota \colon f \longmapsto (23)$



Action equivalence, formally

Definition

Two actions $\phi_1\colon G_1 \longrightarrow \mathsf{Perm}(S_1)$ and $\phi_2\colon G_2 \longrightarrow \mathsf{Perm}(S_2)$ are equivalent if there is an isomorphism $\iota\colon G_1 \to G_2$ and a bijection $\sigma\colon S_1 \to S_2$ such that

$$\sigma \circ \phi_1(g) = \phi_2(\iota(g)) \circ \sigma, \quad \text{for all } g \in G.$$

We say that the resulting action graphs are action equivalent.

If $G_1 = G_2$ and $\iota\colon G \to G$ is the identity map, then S_1 and S_2 are isomorphic as G-sets.

This can be expressed with a commutative diagram:

$$\begin{tikzcd} S_1 \arrow[r, "\phi_1(g)"] \arrow[d, "\sigma"'] & S_1 \arrow[d, "\sigma"] \\ S_2 \arrow[r, "\phi_2(\iota(g))"'] & S_2 \end{tikzcd}$$

Action equivalence can be used to show that in our binary square example, we could have:

- defined $\phi(r)$ to rotate clockwise, and $\phi(f)$ to flip vertically
- used tiles with a and b, rather than 0 and 1
- read from right-to-left, rather than left-to-right, etc.

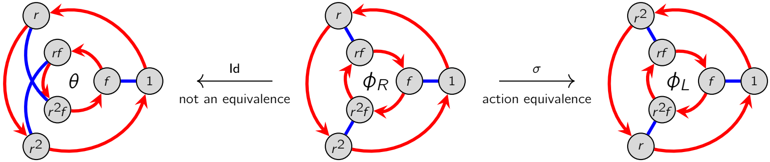
Every right action has an equivalent left action

G acting on...	right action	equivalent left action
itself by multiplication	$x \mapsto xg$	$x \mapsto g^{-1}x$
itself by conjugation	$x \mapsto g^{-1}xg$	$x \mapsto gxg^{-1}$
its subgroups by conjugation	$H \mapsto g^{-1}Hg$	$H \mapsto gHg^{-1}$
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

$$\begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, dashed, "\text{not } \sigma"] \\ x^{-1} \arrow[r, mapsto, "\theta(g)"] & gx^{-1} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, "\sigma"] \\ x^{-1} \arrow[r, mapsto, "\phi_L(g)"] & g^{-1}x^{-1} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\mathrm{Id}"'] & xg \arrow[d, mapsto, dashed, "\text{not } \mathrm{Id}"] \\ x \arrow[r, mapsto, "\theta(g)"] & gx \end{tikzcd}$$

$x \mapsto rx$ — $x \mapsto xr$ — $x \mapsto r^{-1}x = r^2x$

$x \mapsto fx$ — $x \mapsto xf$ — $x \mapsto f^{-1}x = fx$



Every right action has an equivalent left action

G acting on...	right action	equivalent left action
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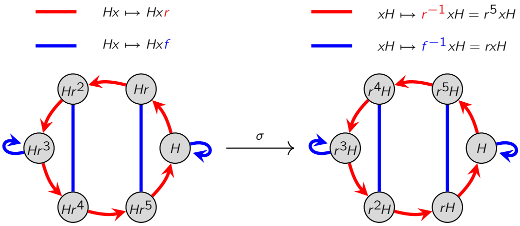
itself by conjugation	$x \mapsto g^{-1}xg$	$x \mapsto gxg^{-1}$
its subgroups by conjugation	$H \mapsto g^{-1}Hg$	$H \mapsto gHg^{-1}$
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

Recall that $aH = bH$ implies $Ha^{-1} = Hb^{-1}$.

$$\begin{tikzcd} Hx \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, "\sigma"'] & Hxg \arrow[d, "\sigma"] \\ x^{-1}H \arrow[r, mapsto, dashed, "\phi_L(g)"] & g^{-1}x^{-1}H \end{tikzcd}$$



Since $aH = bH \not\Rightarrow Ha = Hb$, the the map $xH \mapsto Hx$ is not even well-defined.

Actions by permutations matrices

Consider the following permutation $\pi \in S_5$:

i	1	2	3	4	5
$\pi(i)$	2	3	1	5	4

$$\begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[r, bend left] & 3 \arrow[ll, bend left] & 4 \arrow[r, bend left] & 5 \arrow[l, bend left] \end{tikzcd}$$

$$\pi = (1\,2\,3)\,(4\,5)$$

The permutation matrix P_π permutes the entries of a column vector as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_5 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ x_{\pi(3)} \\ x_{\pi(4)} \\ x_{\pi(5)} \end{bmatrix},$$

and the entries of a row vector as

$$\begin{aligned} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} &= \begin{bmatrix} x_2 & x_3 & x_1 & x_5 & x_4 \end{bmatrix} \\ &= \begin{bmatrix} x_{\pi^{-1}(1)} & x_{\pi^{-1}(2)} & x_{\pi^{-1}(3)} & x_{\pi^{-1}(4)} & x_{\pi^{-1}(5)} \end{bmatrix}. \end{aligned}$$

Actions by permutations matrices

In general, a left action of S_n on a set of vectors X

$$\phi_L \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_L(\pi) \colon x \longmapsto P_\pi x$$

is equivalent to the right action

$$\phi_R \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_R(\pi) \colon x \longmapsto x^T P_\pi^T = x^T P_{\pi^{-1}}$$

via the transpose map.

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_L(\pi)"] \arrow[d, mapsto, "T"'] & P_\pi x \arrow[d, mapsto, "T"] \\
x^T \arrow[r, mapsto, dashed, "\phi_R(\pi)"] & x^T P_{\pi^{-1}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \arrow[r, mapsto, "P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}"] \arrow[d, mapsto, "T"'] & \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} \arrow[d, mapsto, "T"] \\
\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \arrow[r, mapsto, dashed, "P_{\pi^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P^T"] & \begin{bmatrix} x_3 & x_1 & x_2 \end{bmatrix}
\end{tikzcd}$$

Another equilalence between left and right actions of permutations

Recall the two "canonical" ways label a Cayley graph for $S_3 = \langle (12), (23) \rangle$ with the set

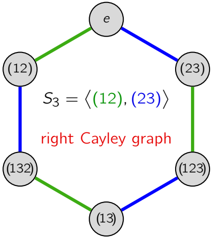
$$X = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}.$$

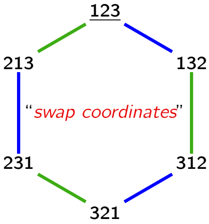
In one, (ij) can be interpreted to mean

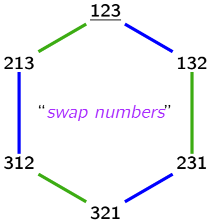
"*swap the numbers in the i^{th} and j^{th} coordinates*."

Alternatively, (ij) could mean

"*swap the numbers i and j, regardless of where they are*."







One of these is a right group action, and the other a left group action

Another equilalence between left and right actions of permutations

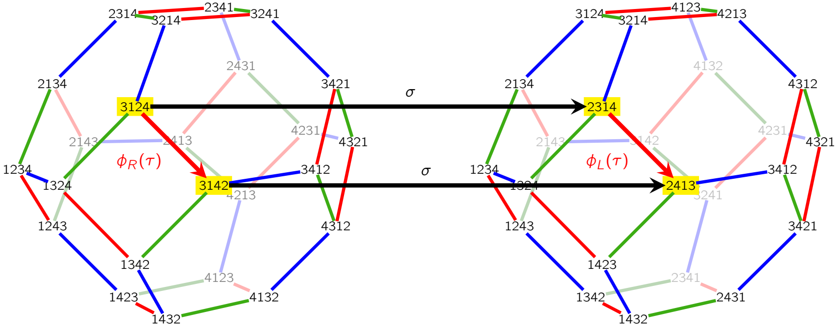
i	1 2 3 4
π	**3 1 2 4**

i	1 2 3 4
$\pi\tau$	**3 1 4 2**

$$\begin{tikzcd}
\pi(1)\pi(2)\pi(3)\pi(4) \arrow[r, mapsto, "\phi_R(\tau)", "\textit{swap coordinates 3 and 4}"'] \arrow[d, mapsto, "\sigma"'] & \pi(1)\pi(2)\pi(4)\pi(3) \arrow[d, mapsto, "\sigma"] \\
\pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(3)\pi^{-1}(4) \arrow[r, mapsto, "\phi_L(\tau)", "\textit{swap digits 3 and 4}"'] & \pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(4)\pi^{-1}(3)
\end{tikzcd}$$

i	1 2 3 4
π^{-1}	**2 3 1 4**

i	1 2 3 4
$\tau^{-1}\pi^{-1}$	**2 4 1 3**



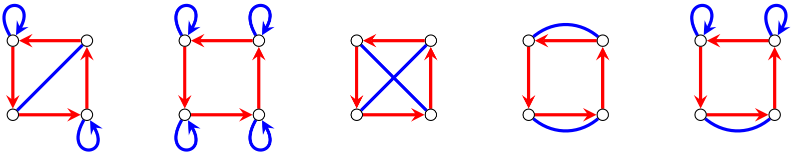
"*swap coordinates*" "*swap numbers*"

Classification of G-sets

Natural question

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi \colon G \to \mathrm{Perm}(S)$ is

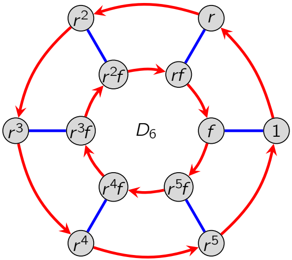
- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathrm{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathrm{Ker}(\phi) = \langle e \rangle$. ("*no broken buttons, except* $1 \in G$")

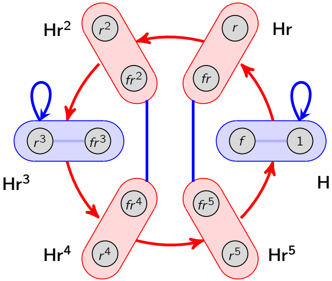
In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

Proposition

Every transitive G-action is equivalent to G acting on a set of cosets by multiplication.

A connected action graph is a Cayley graph collapsed by right cosets of some subgroup.

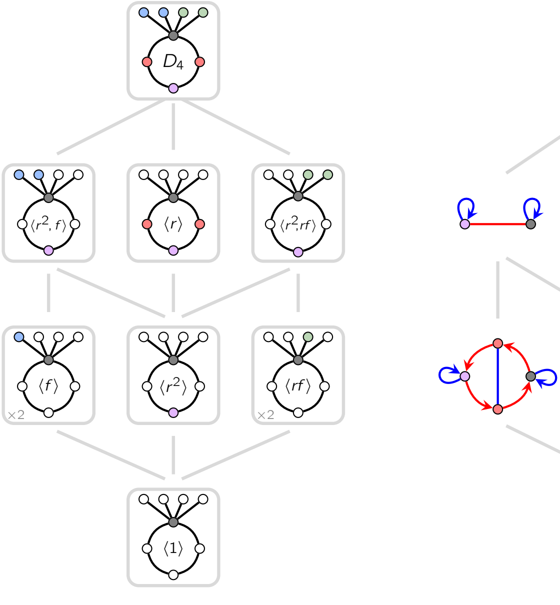




collapse right cosets of H (an action)

We can *always* collapse by right cosets. We can collapse by left cosets iff H is normal.

The transitive D_4-sets: collapsing by right cosets



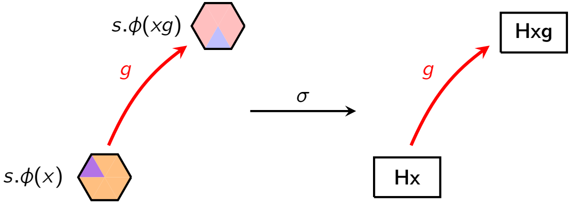
Transitive actions

Proposition

Every transitive G-action is equivalent to G acting on a set of cosets by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix $s \in S$, let $H = \mathrm{stab}(s)$, and define

$$\sigma\colon S \longrightarrow H\backslash G, \qquad \sigma\colon s.\phi(x) \longmapsto Hx$$



Show that σ is a well-defined bijection, and then the proof follows because:

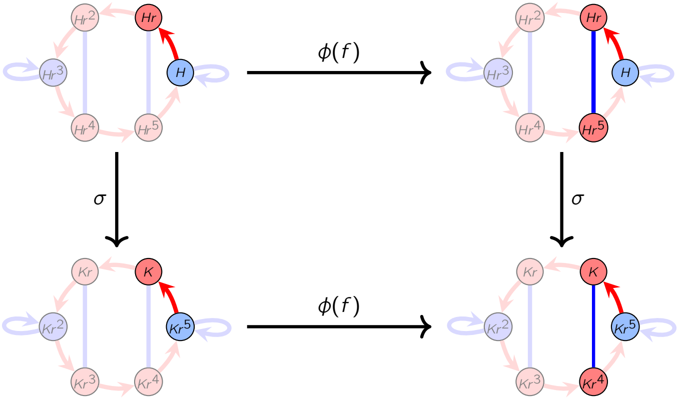
$$\begin{tikzcd} S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ H\backslash G \arrow[r, "\psi(g)"'] & H\backslash G \end{tikzcd} \qquad \begin{tikzcd} s.\phi(x) \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & s.\phi(xg) \arrow[d, "\sigma"] \\ Hx \arrow[r, mapsto, "\psi(g)"'] & Hxg \end{tikzcd}$$

Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

Consider $H = \langle f \rangle$ and $K = r^{-1}Hr = \langle r^4 f \rangle$. Define $\sigma \colon Hx \mapsto Kr^{-1}x$.



Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then the G-sets $H\backslash G$ and $K\backslash G$ are isomorphic.

Proof

Define the map

$$\sigma\colon H\backslash G \longrightarrow K\backslash G, \qquad \sigma\colon Hx \longmapsto Ka^{-1}x.$$

We claim that this is a well-defined bijection, and commutes with $\phi(g)$:

$$\begin{tikzcd} H\backslash G \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & H\backslash G \arrow[d, "\sigma"] \\ K\backslash G \arrow[r, "\phi(g)"'] & K\backslash G \end{tikzcd} \qquad \begin{tikzcd} Hx \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & Hxg \arrow[d, "\sigma"] \\ Ka^{-1}x \arrow[r, mapsto, "\phi(g)"'] & Ka^{-1}xg \end{tikzcd}$$

<u>Well-defined</u>: Suppose $Hx = Hy$. Then $Hyx^{-1} = H$, so $yx^{-1} \in H$.

$$\sigma(Hx) = Ka^{-1}x = \underbrace{a^{-1}H}_{=Ka^{-1}}x = a^{-1}\underbrace{(Hyx^{-1})}_{=H}x = \underbrace{a^{-1}H}_{=Ka^{-1}}y = Ka^{-1}y = \sigma(Hy).$$

Conjugates of stab(s) give the same G-set

Proposition

If $K=a^{-1}Ha$, then the G-sets $H\backslash G$ and $K\backslash G$ are isomorphic.

Proof

Define the map
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Injectivity: Suppose $\sigma(Hx)=\sigma(Hy)$. Then

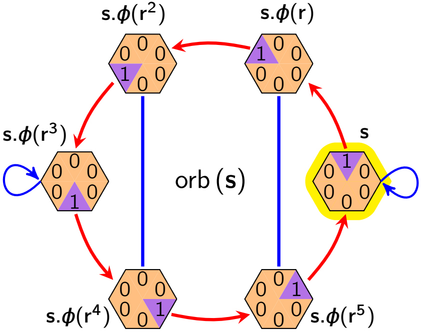
$$\sigma(Hx)=\underbrace{Ka^{-1}}_{=a^{-1}H}x=a^{-1}Hx,\qquad\text{and}\qquad \sigma(Hy)=\underbrace{Ka^{-1}}_{=a^{-1}H}y=a^{-1}Hy,$$

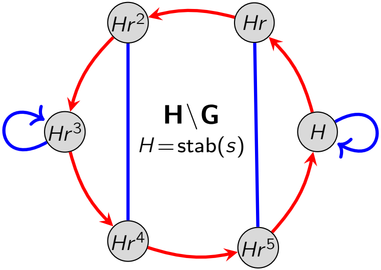
and thus $Hx=Hy$. Surjectivity is straightforward. $\square$

Transitive actions

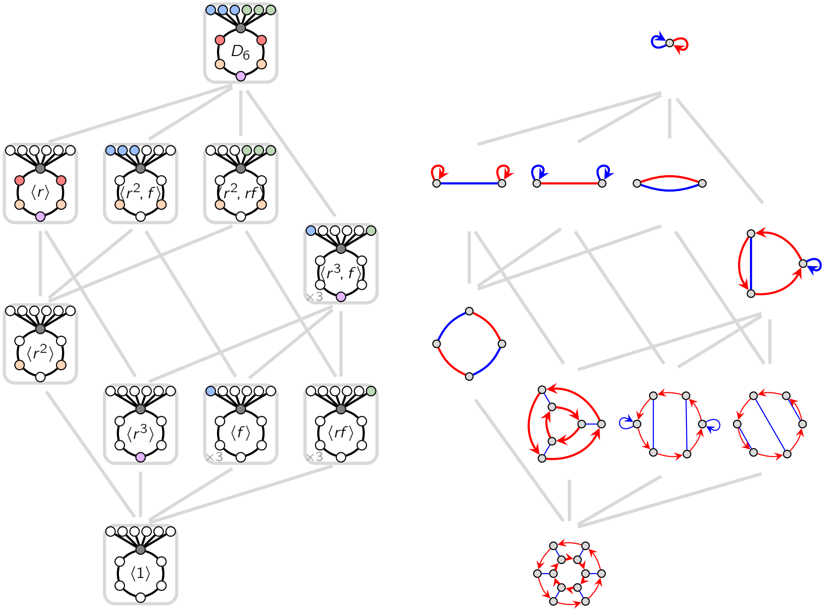
Big ideas

- Every transitive G-action is isomorphic to G acting on the cosets of **stab**(s).
- The action graph is constructed by collapsing by right cosets of **stab**(s).
- conjugates of **stab**(s) give the same G-set.





The transitive D_6-sets: collapsing by right cosets

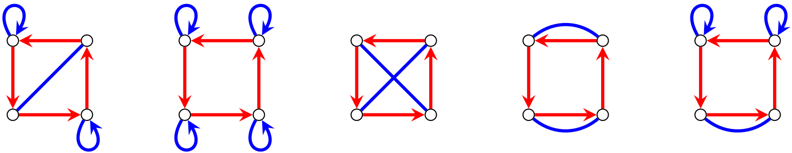


Classification of G-sets

Natural question

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi\colon G \to \mathrm{Perm}(S)$ is

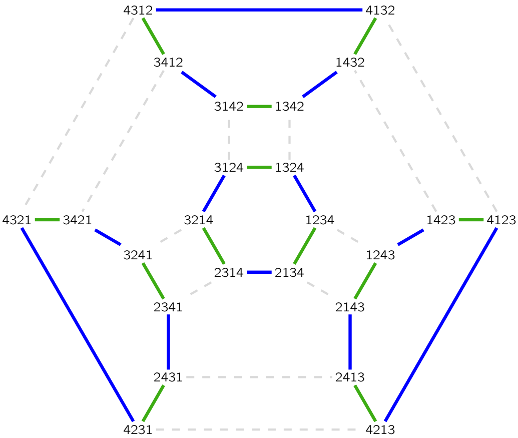
- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathrm{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathrm{Ker}(\phi) = \langle e \rangle$. ("*no broken buttons, except* $1 \in G$")

In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

An example of a free action that is not transitive

The group $S_3 = \langle (12), (23) \rangle$ acts on permutations **1234**, via $\phi\colon S_3 \to \mathsf{Perm}(S)$, where

- $\phi((12))$ = the permutation that swaps the 1st and 2nd coordinates
- $\phi((23))$ = the permutation that swaps the 2nd and 3rd coordinates

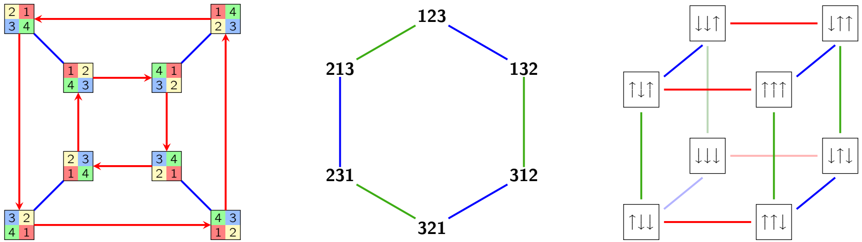


Simply transitive actions

Definition

An action $\phi\colon G \to \mathrm{Perm}(S)$ is simply transitive if it is transitive and free.

Here are some simply transitive actions that we have seen.



Proposition

Every simply transitive G-action is equivalent to G acting on itself by multiplication.

This just says that *simply transitive G-sets are groups*!

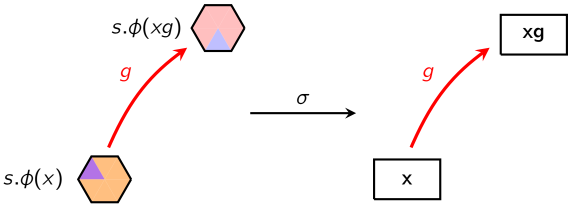
Simply transitive actions

Proposition

Every simply transitive G-action is equivalent to G acting on itself by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix our "home state" $s \in S$, and define

$$\sigma\colon S \longrightarrow G, \qquad \sigma\colon s.\phi(x) \longmapsto x$$



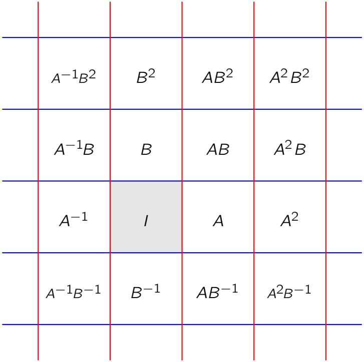
Show that σ is a well-defined bijection, and then the proof follows because:

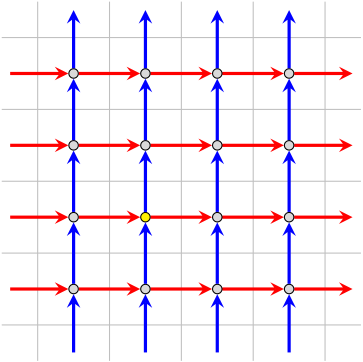
$$\begin{tikzcd} S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ G \arrow[r, "\psi(g)"'] & G \end{tikzcd} \qquad \begin{tikzcd} s.\phi(x) \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & s.\phi(xg) \arrow[d, "\sigma"] \\ x \arrow[r, mapsto, "\psi(g)"'] & xg \end{tikzcd}$$

Simply transitive actions from reflection groups

One place where simply transitive actions arise is from tilings.

The group $\langle A, B \mid AB = BA \rangle \cong \mathbb{Z} \times \mathbb{Z}$ acts simply transitively on the unit squares in $\mathbb{Z}^2$.

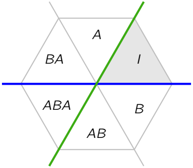




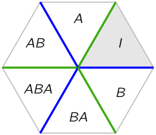
The shaded region is called a **fundamental chamber**.

Simply transitive actions from finite reflection groups

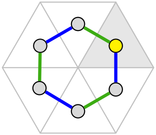
The dihedral group $D_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = 1 \rangle$ acts simply transitively on the six regions of a hexagon.



"*left action*"

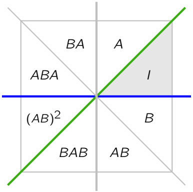


"*right action*"

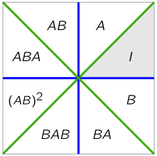


"*(right) Cayley graph*"

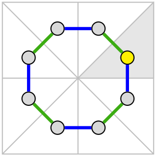
The dihedral group D_4 acts simply transitively on the eight regions of a square.



"*left action*"



"*right action*"

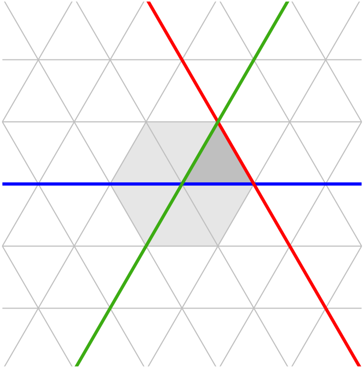


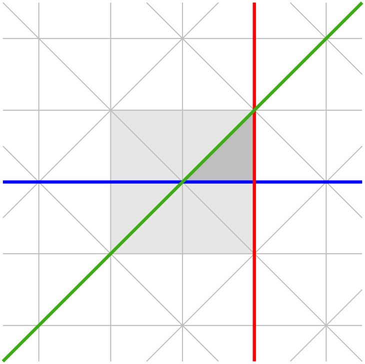
"*(right) Cayley graph*"

Simply transitive actions from affine reflection groups

In both previous examples, adding a third reflection generates a tiling of the plane.

The resulting affine groups, $\mathsf{Aff}(D_3)$ and $\mathsf{Aff}(D_4)$, act simply transitively on the chambers.

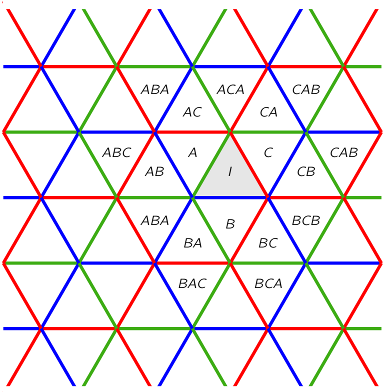


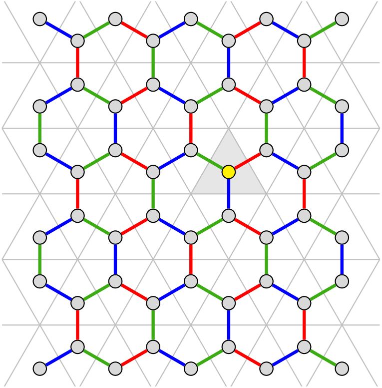


Simply transitive actions and affine Weyl groups

The group $\mathrm{Aff}(D_3)$ is better known as the affine Weyl group of type A_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





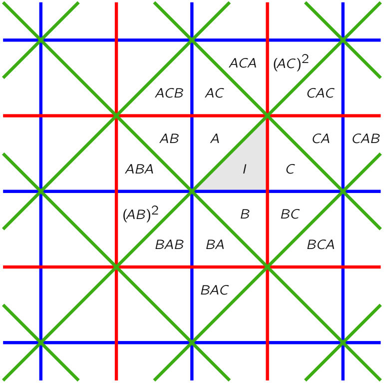
It has presentation

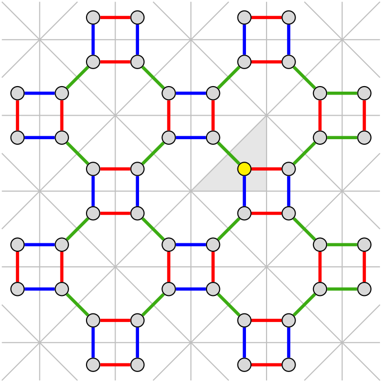
$$W(\tilde{A}_2) = \mathrm{Aff}(D_3) = \left\langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^3 = (AC)^3 = (BC)^3 = 1 \right\rangle.$$

Simply transitive actions and affine Weyl groups

The group $\mathrm{Aff}(D_4)$ is better known as the affine Weyl group of type C_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





It has presentation

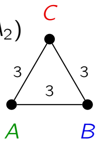
$$W(\tilde{C}_2) = \mathrm{Aff}(D_4) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^4 = (AC)^4 = (BC)^2 = 1 \rangle.$$

Weyl groups and Dynkin diagrams

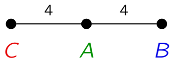
The presentations of the affine Weyl groups are encoded by Dynkin diagrams.

Nodes s_i are generators, and the labeled edges m_{ij} describe relations: $(s_i s_j)^{m_{ij}} = 1$.

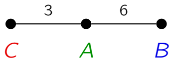
$\mathrm{Aff}(D_3) = W(\tilde{A}_2)$



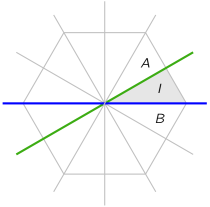
$\mathrm{Aff}(D_4) = W(\tilde{C}_2)$

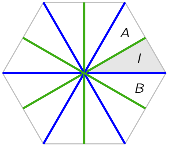


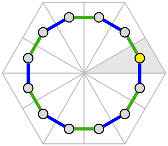
$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



This last example is the affine version of $D_6 = \langle A, B \mid A^2 = B^2 = (AB)^6 = 1 \rangle$ acting simply transitively on the 12 regions of a hexagon.



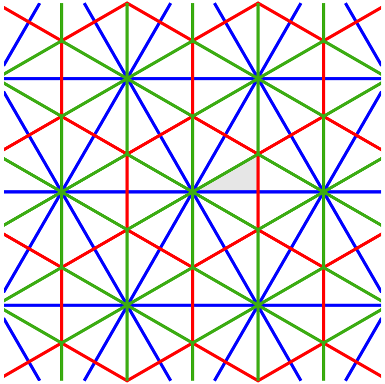


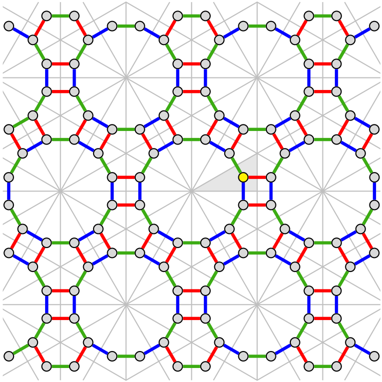


One last affine Weyl group

The group $\mathrm{Aff}(D_6)$ is better known as the the affine Weyl group of type G_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





It has presentation

$$W(\tilde{G}_2) = \mathrm{Aff}(D_6) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^6 = (AC)^3 = (BC)^2 = 1 \rangle.$$

Coxeter groups and tilings of hyperbolic space

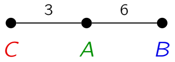
A Coxeter group is a group generated by "reflections", with presentation

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle.$$

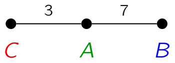
Like Weyl groups, this can be encoded by a Coxeter graph.

Some Coxeter groups act simply transitively on chambers of hyperbolic tilings.

$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



A hyperbolic Coxeter group



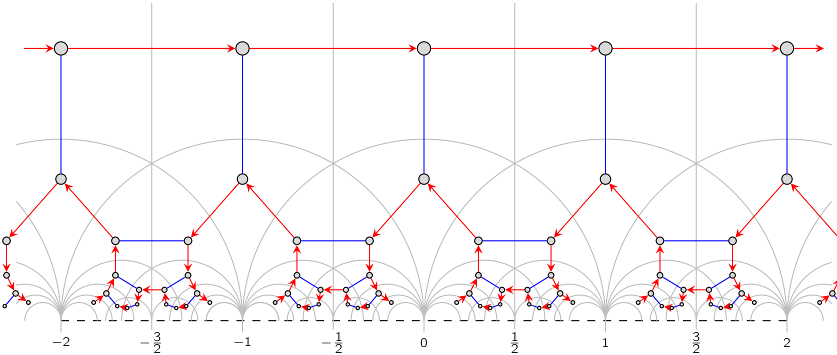
A simply transitive action of $\mathrm{PSL}_2(\mathbb{Z})$

The projective special linear group

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\langle -I \rangle, \qquad \text{where } \mathrm{SL}_2(\mathbb{Z}) = \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{T} \right\rangle$$

defines a tiling of hyperbolic ideal triangles in the upper half-plane via

$$S\colon z \longmapsto \frac{0z-1}{z+0} = -\frac{1}{z}, \qquad \text{and} \qquad T\colon z \longmapsto \frac{z+1}{0z+1} = z+1,$$



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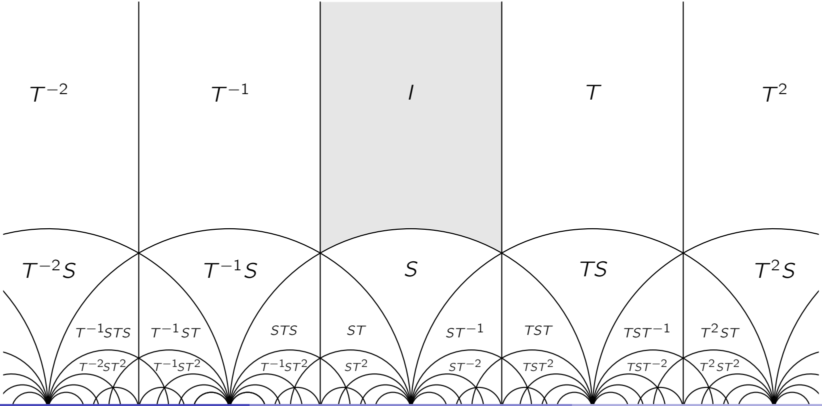


Equivariant maps

Definition

Suppose G acts on S_i via $\phi_i \colon G \to \mathrm{Perm}(S_i)$ for $i = 1, 2$. A **G-equivariant map** is a function $\sigma \colon S_1 \to S_2$ such that $\sigma \circ \phi_1(g) = \phi_2(g) \circ \sigma$, for all $g \in G$:

$$\begin{tikzcd} S_1 \arrow[r, mapsto, "\phi_1(g)"] \arrow[d, mapsto, "\sigma"'] & S_1 \arrow[d, mapsto, "\sigma"] \\ S_2 \arrow[r, mapsto, "\phi_2(g)"] & S_2 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r, mapsto, "\phi_1(g)"] \arrow[d, mapsto, "\sigma"'] & s_1.\phi_1(g) \arrow[d, mapsto, "\sigma"] \\ s_2 \arrow[r, mapsto, "\phi_2(g)"] & s_2.\phi_2(g) \end{tikzcd}$$

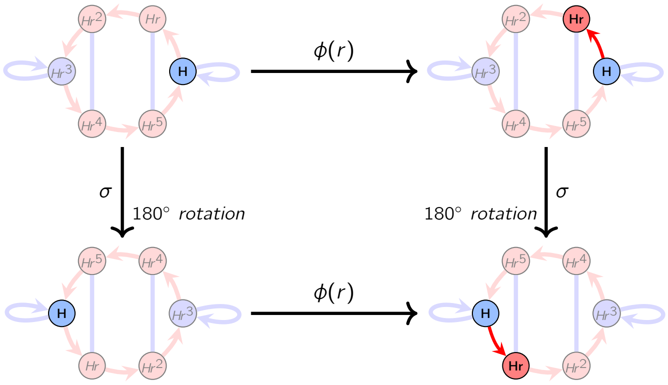
Loosely speaking, "*equivariance*" means "*commutes with the action*."

Key concepts

- Action equivalence involves an isomorphism $\iota \colon G_1 \to G_2$ and bijection $\sigma \colon S_1 \to S_2$.
- G-set isomorphisms ($\iota = \mathrm{Id}$, σ is bijective) are equivariant bijections.
- G-set automorphisms ($\iota = \mathrm{Id}$, σ is bijective, $S_1 = S_2$) form a group $\mathrm{Aut}_G(S)$.
- G-set homomorphisms ($\iota = \mathrm{Id}$, but σ need not be bijective) are equivariant maps.

G-set automorphisms as symmetries of the action graph

Let $S = G\backslash H$, for $G = D_6$ and $H = \langle f \rangle$.

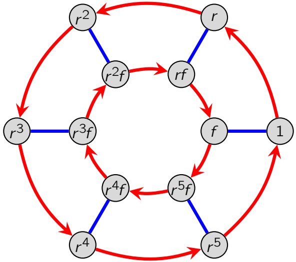


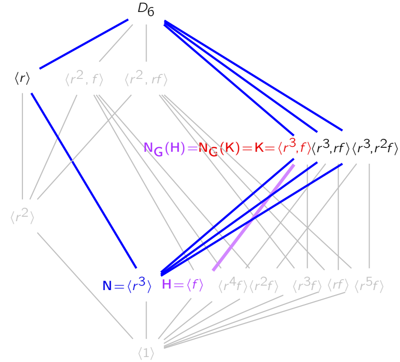
Key idea

The action and bijection clearly commute upon thinking of:

- the action $\phi(r)$ as right-multiplying Hr^i by r,
- the bijection σ as left-multiplying Hr^i by r^3. (This works because $r^3 \in N_G(H)$.)

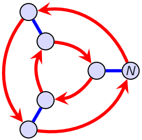
G-set automorphisms





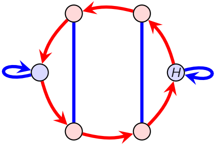
What do you notice about normalizers vs. symmetries of the actions graphs?

$N = \langle r^3 \rangle$; *normal*



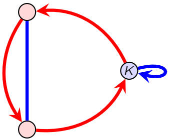
$\mathrm{Aut}_G(N\backslash G) \cong D_3 \cong N_G(N)/N$

$H = \langle f \rangle$; *moderately unnormal*



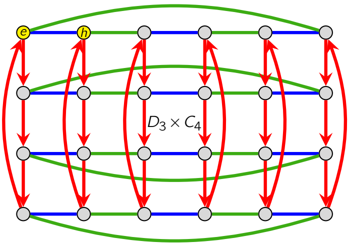
$\mathrm{Aut}_G(H\backslash G) \cong C_2 \cong N_G(H)/H$

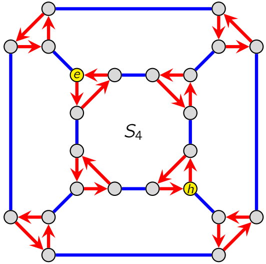
$K = \langle r^3, f \rangle$; *fully unnormal*

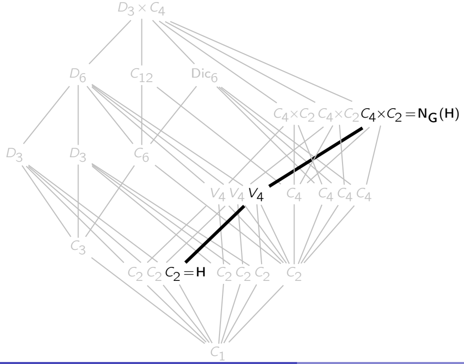


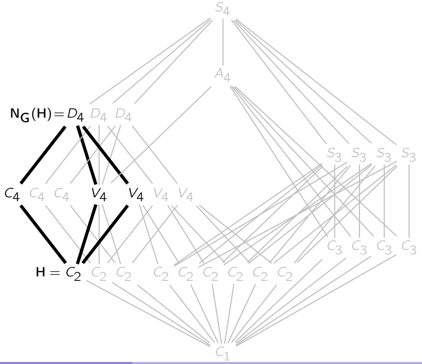
$\mathrm{Aut}_G(K\backslash G) \cong \langle 1 \rangle \cong N_G(K)/K$

G-set automorphisms

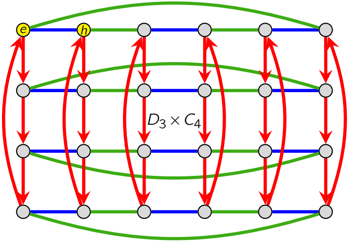


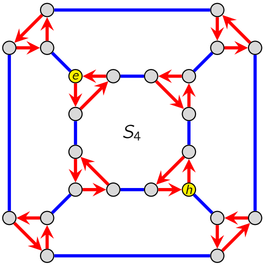






G-set automorphisms



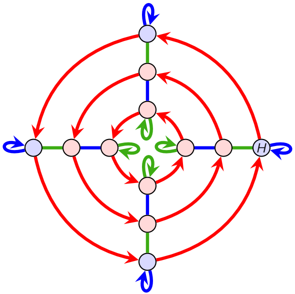


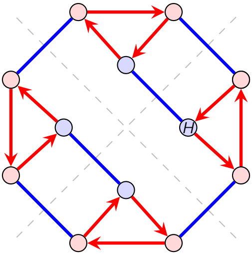
$H = \langle (f,1) \rangle \leq D_3 \times C_4 = G$

$\mathrm{Aut}_G(H\backslash G) \cong N_G(H)/H \cong C_4$

$H = \langle ((12)(34)) \rangle \leq S_4 = G$

$\mathrm{Aut}_G(H\backslash G) \cong N_G(H)/H \cong V_4$





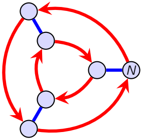
The G-set automorphism group

Theorem

For any $H \leq G$, the G-set automorphism group of $S = H\backslash G$ is

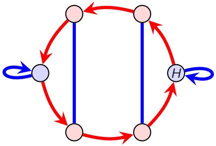
$$\mathrm{Aut}_G(S) \cong N_G(H)/H.$$

$N = \langle r^3 \rangle$; *normal*



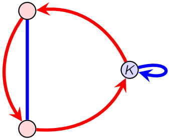
$\mathrm{Aut}_G(N\backslash G) \cong D_3$

$H = \langle f \rangle$; *moderately unnormal*



$\mathrm{Aut}_G(H\backslash G) \cong C_2$

$K = \langle r^3, f \rangle$; *fully unnormal*



$\mathrm{Aut}_G(K\backslash G) \cong C_1$

Here's how the proof will go, given $\sigma \in \mathrm{Aut}_G(S)$, and $S = H\backslash G$:

1. **Lemma 1**: $\sigma\colon Hg \mapsto Hxg$, for some fixed $x \in G$ (call this σ_x).
2. **Lemma 2**: $\sigma_x \in \mathrm{Aut}_G(S)$ iff $x \in N_G(H)$. That is, $\sigma_x\colon Hg \mapsto xHg$.
3. **FHT**: Two $\sigma_x = \sigma_{x'}$ iff x, x' are in the same coset of H.

The G-set automorphism group

Lemma 1

Any G-set automorphism $\sigma \in \mathrm{Aut}_G(S)$, for $S = H\backslash G$, is determined by the image of H:

$$\text{if } \sigma\colon H \mapsto Hx, \quad \text{then} \quad \sigma\colon Hg \mapsto Hxg, \text{ for all } g \in G.$$

Proof

Since σ is G-equivariant, it commutes with each $\phi(g) \in \mathrm{Perm}(S)$.

That is, the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ S \arrow[r, "\phi(g)"] & S \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & Hg \arrow[d, mapsto, "\sigma"] \\ Hx \arrow[r, mapsto, "\phi(g)"] & Hxg \end{tikzcd}$$

It follows that $\sigma\colon Hg \mapsto Hxg$, as claimed. $\square$

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"$\Rightarrow$": Suppose $\sigma_x \in \mathrm{Aut}_G(H\backslash G)$, and take $h \in H$. We have:

$$\begin{tikzcd} S \arrow[r, "\phi(h)"] \arrow[d, "\sigma_x"'] & S \arrow[d, "\sigma_x"] \\ S \arrow[r, "\phi(h)"] & S \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, mapsto, "\phi(h)"] \arrow[d, mapsto, "\sigma_x"'] & H \arrow[d, mapsto, "\sigma_x"] \\ Hx \arrow[r, mapsto, "\phi(h)"] & Hxh = Hx \end{tikzcd}$$

That is, for every $h \in H$,

$$H = Hxhx^{-1} \quad \Leftrightarrow \quad xhx^{-1} \in H \quad \Leftrightarrow \quad x \in N_G(H). \qquad \checkmark$$

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"$\Leftarrow$": Suppose $x \in N_G(H)$, and pick $g \in G$.

By Lemma 1: $\sigma_x \colon Hg \mapsto Hxg = xHg$.

The operations σ_x (left-multiplying by x), and $\phi(g)$ (right-multiplying by g) clearly commute. ✓

$$\begin{tikzcd}
S \arrow[r, "\phi(g)"] \arrow[d, "\sigma_x"'] & S \arrow[d, "\sigma_x"] \\
S \arrow[r, "\phi(g)"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\phi(x)"'] & Hg \arrow[d, mapsto, "\phi(x)"] \\
Hx \arrow[r, mapsto, "\phi(g)"] & Hxg = xHg
\end{tikzcd}$$

The G-set automorphism group

Theorem

If G acts on the set $S = H\backslash G$ of right cosets of $H \leq G$, then

$$\mathrm{Aut}_G(S) \cong N_G(H)/H.$$

Proof

We'll apply the FHT to the map

$$f \colon N_G(H) \longrightarrow \mathrm{Aut}_G(S), \qquad x \longmapsto \sigma_x,$$

where $\sigma_x \colon Hg \longmapsto Hxg$.

Homomorphism: Straightforward exercise. ✓

Onto: Immediate from Lemma 2. ✓

$\mathrm{Ker}(f) = H$. "$\subseteq$":

$$x \in \mathrm{Ker}(\phi) \quad \Leftrightarrow \quad Hg = Hxg,\ \forall\, g \in G \quad \Leftrightarrow \quad H = Hx \quad \Leftrightarrow \quad x \in H.$$

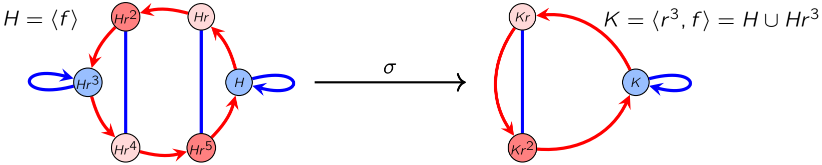
"$\supseteq$": If $h \in H$, then $\sigma_h \colon Hg \mapsto Hhg = Hx$. ✓

The result now follows from the FHT. □

G-set homomorphisms

Dropping bijectivity of $\sigma\colon S_1 \to S_2$ defines a G-set homomorphism, or G-equivariant map.

Consider this example of D_6-sets:



This can be described by the following commutative diagram:

$$
\begin{tikzcd}
H\backslash G \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & H\backslash G \arrow[d, mapsto, "\sigma"] \\
K\backslash G \arrow[r, mapsto, "\phi(g)"] & K\backslash G
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Hx \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & Hxg \arrow[d, mapsto, "\sigma"] \\
Kx \arrow[r, mapsto, "\phi(g)"] & Kxg
\end{tikzcd}
$$

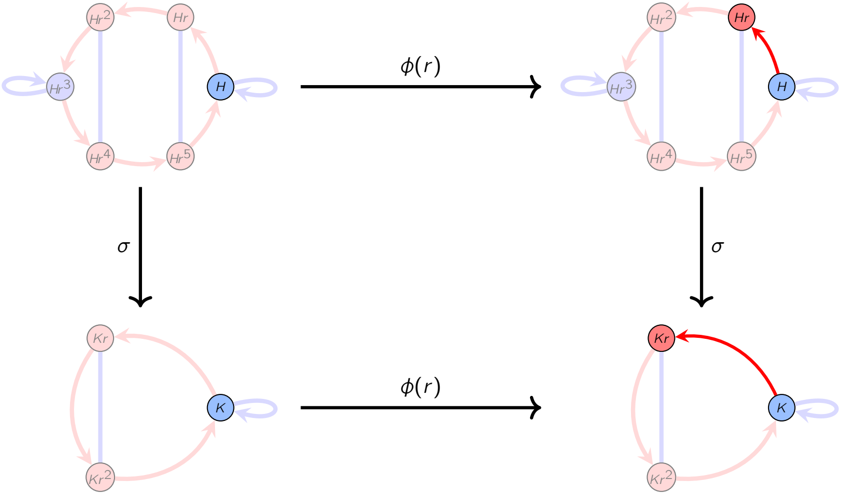
Key idea

We say that "*the map σ commutes with the action of the group*."

G-set homomorphisms

Here is that example again for $G = D_6$ and subgroups:

$H = \langle f \rangle$ (*moderately unnormal*), $\qquad K = \langle r^3, f \rangle = H \cup Hr^3 = N_G(H)$, (*fully unnormal*)

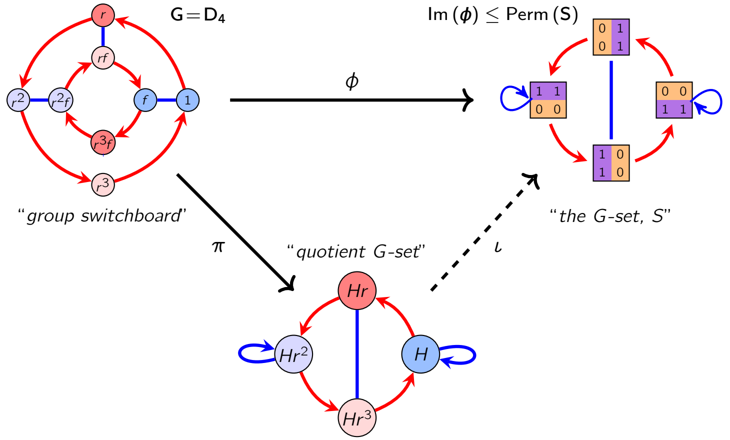


The "fundamental homomorphism theorem for G-sets"

Orbit-stabilizer theorem, restated

If $\phi\colon G \to \mathsf{Perm}(S)$ is a transitive action and $s \in S$, then $\mathsf{orb}(s)$ is isomorphic to the quotient of G by $H = \mathsf{stab}(s)$:

$$H \backslash G \cong \mathsf{orb}(s).$$



Group presentations, formalized

Every group $G = \langle a, b \rangle$ satisfying $a^4 = 1$, $b^2 = 1$, and $(ab)^2 = 1$ is a quotient of D_4.