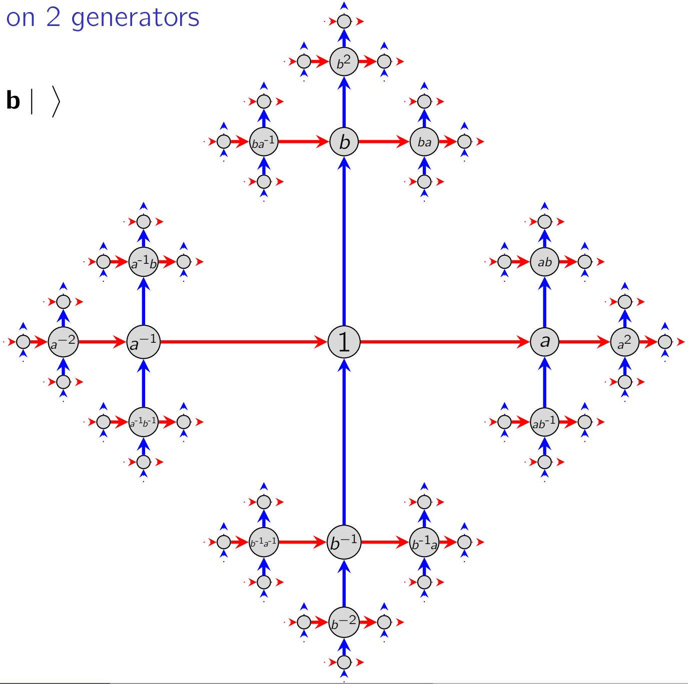
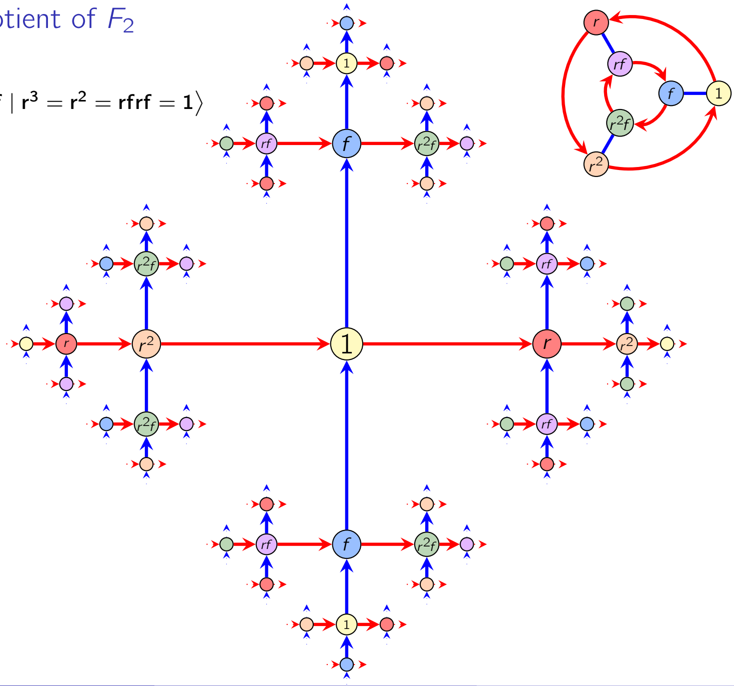
The free group on 2 generators

$$\mathbf{F_2} = \langle \mathbf{a}, \mathbf{b} \mid \ \rangle$$



D_3 as a quotient of F_2

$$\mathbf{D_3} = \langle \mathbf{r}, \mathbf{f} \mid \mathbf{r}^3 = \mathbf{r}^2 = \mathbf{rfrf} = \mathbf{1} \rangle$$



Factoring maps

A number of properties can be described as

"*the minimal* ____," or "*the maximal* ____,"

satisfying some condition.

We can express this concisely in terms of maps and commutative diagrams.

This highlights similarities and patterns that are inherent in seemingly different structures, streamline proofs, and leads to new insight.

Warm-up exercise (easy)

Given maps $g\colon G \to H$ and $h\colon H \to K$, their composition $f := h \circ g$ is a map from G to K.

I.e., there is *always* a map $f\colon G \to K$ making the following diagram commute:

$$\begin{tikzcd} G \arrow[rr, dashed, "f = h \circ g"] \arrow[dr, "g"'] & & K \\ & H \arrow[ur, "h"'] & \end{tikzcd}$$

Factoring maps

Let's now consider two variants of the previous commutative diagram.

Definition

Given two maps. . .

1. *from the same domain*, $f: G \to K$, $g: G \to H$, when does there exist $h: H \to K$
2. *into the same codomain*, $f: G \to K$, $h: H \to K$, when does there exist $g: G \to H$

such that $f = h \circ g$?

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "f"] \arrow[dr, "g"'] & & K \\ & H \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

"f factors through g"

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, dashed, "g"'] & & \mathbf{K} \\ & H \arrow[ur, "h"'] & \end{tikzcd}$$

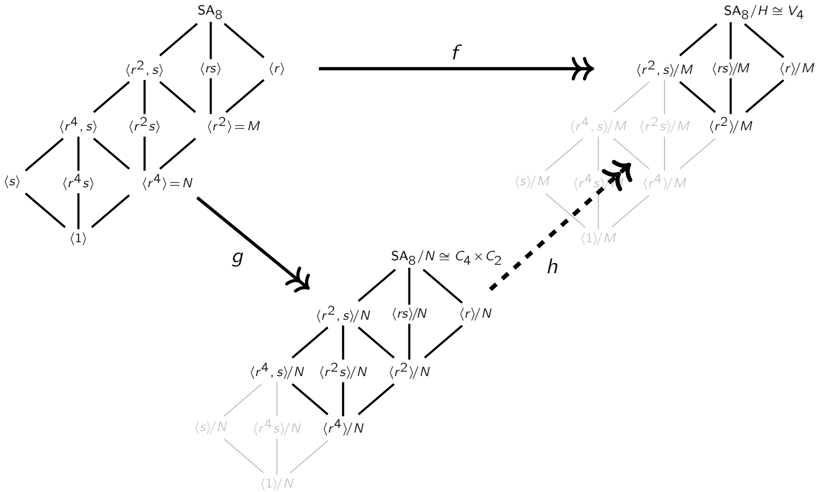
"f factors through h"

We say that h and g are **factors** of f.

Factoring maps: a quotient between the codomains

Let $G = \mathrm{SA}_8 = \langle r, s\rangle$, and $N = \langle r^2\rangle \cong C_4$, and $M = \langle r^4\rangle \cong C_2$.

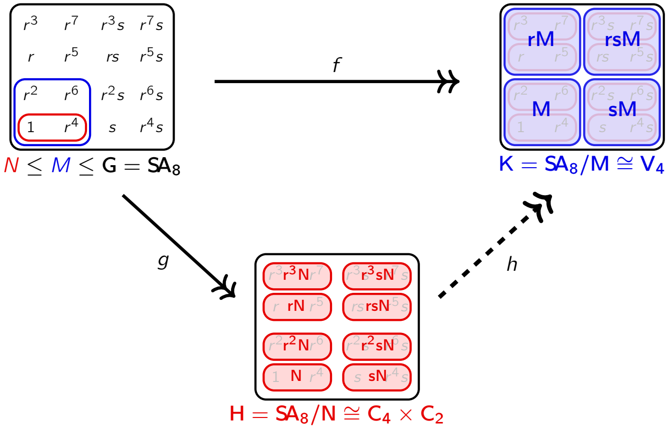
The standard quotient map $f\colon \mathrm{SA}_8 \to V_4$ can be factored:



Factoring maps: a quotient between the codomains

Formally, this map is defined by

$$h\colon \ \mathrm{SA}_8\,/N \longrightarrow \mathrm{SA}_8\,/M, \qquad h\colon gN \longmapsto gM.$$



Factoring maps: a quotient between the codomains

Formally, this map is defined by

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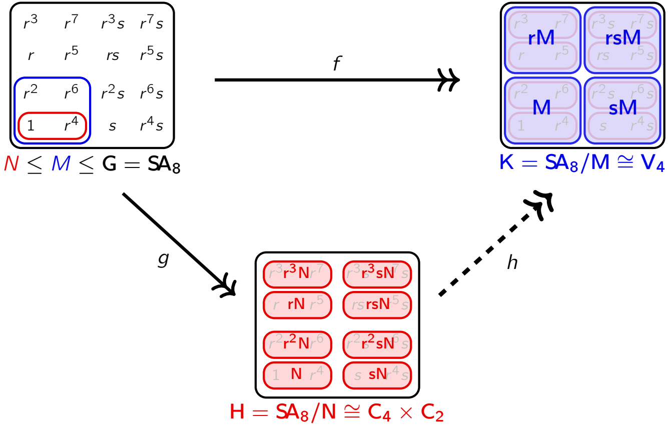


Canceling maps: when does existence imply uniqueness?

Proposition ("cancelation laws")

Suppose we have functions $g_i \colon G \to H$ and $h_i \colon H \to K$ between sets, for $i = 1, 2$.

If g is surjective, then it right-cancels: $\quad h_1 \circ g = h_2 \circ g \quad \Longrightarrow \quad h_1 = h_2.$

$$\begin{tikzcd} G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "g"'] & & K \\ & H \arrow[ur, dashed, two heads, "h_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "g"'] & & K \\ & H \arrow[ur, dashed, two heads, "h_2"'] & \end{tikzcd} \quad \Longrightarrow \quad h_1 = h_2.$$

If h is injective, then it left-cancels: $\quad h \circ g_1 = h \circ g_2 \quad \Longrightarrow \quad g_1 = g_2.$

$$\begin{tikzcd} G \arrow[rr, hook, "f"] \arrow[dr, dashed, hook, "g_1"'] & & K \\ & H \arrow[ur, hook, "h"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[rr, hook, "f"] \arrow[dr, dashed, hook, "g_2"'] & & K \\ & H \arrow[ur, hook, "h"'] & \end{tikzcd} \quad \Longrightarrow \quad g_1 = g_2.$$

Key idea

Injective functions have left inverses; surjective functions have right inverses.

Motivating the co-universal property of quotient groups

Definition

Given $H \le G$, the **canonical inclusion map** is

$$\iota\colon H \hookrightarrow G, \qquad \iota\colon h \longmapsto h.$$

If $H \trianglelefteq G$, the **canonical quotient map is**

$$\pi\colon G \longrightarrow G/H, \qquad \pi\colon g \longmapsto gH.$$

There does not exist a homomorphism $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_4$ with $\phi(1) = 1$. To formalize this:

the canonical quotient $f: \mathbb{Z} \twoheadrightarrow \mathbb{Z}_4$ *does not factor through* $g\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}_3$.

That is, there does *not* exist $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_4$ making this diagram commute:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\pi"'] & & \mathbb{Z}_4 \\
& \mathbb{Z}_3 \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & 1 \\
& 1 \arrow[ur, dashed, mapsto, "\phi"'] &
\end{tikzcd}$$

Preview: such a map exists iff $\mathrm{Ker}(\pi) \le \mathrm{Ker}(f)$, i.e., f collapses at least as much as π.

Motivating the co-universal property of quotient groups

Does $\phi\colon \mathbb{Z}_8 \to \mathbb{Z}_{12}$, where $\phi(1) = 3$, define a homomorphism?

Is there a homomorphism ϕ making the following diagram commute?

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\pi"'] & & \mathbb{Z}_{12} \\
& \mathbb{Z}_8 \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & 3 \\
& 1 \arrow[ur, mapsto, dashed, "\phi"'] &
\end{tikzcd}$$

Note that $\mathrm{Ker}(f) = 4\mathbb{Z}$ is a subgroup of $\mathrm{Ker}(\pi) = 8\mathbb{Z}$, and so f factors through π.

Not only does ϕ exist, it is automatically unique by the cancellation laws.

The co-universal property of quotient groups

Theorem

Let $N \trianglelefteq G$ and $f\colon G \to K$ be a homomorphism such that $N \leq \mathrm{Ker}(f)$. Then

1. f uniquely factors through $\pi\colon G \to G/N$ (i.e., $\exists! h\colon G/N \to K$ such that $g = h \circ \pi$).
2. h is injective iff $\mathrm{Ker}(f) = N$.

Proof (i)

Assume WLOG that f is onto (otherwise, take $K = \mathrm{Im}(f)$). Define $h\colon G/N \to H$ by

$$\begin{tikzcd} G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\pi"'] & & K \\ & G/N \arrow[ur, dashed, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & f(x) \\ & xN \arrow[ur, dashed, mapsto, "h"'] & \end{tikzcd}$$

Well-defined: If $xN = yN$, then $y^{-1}xN = N$, so $y^{-1}x \in N = \mathrm{Ker}(\pi) \leq \mathrm{Ker}(f)$. Now,

$$f(y^{-1}x) = 1 \implies f(y)^{-1}f(x) = 1 \implies h(xN) = f(x) = f(y) = h(yN). \qquad \checkmark$$

Homomorphism: $h(xNyN) = h(xyN) = f(xy) = f(x)f(y) = h(xH)h(yN).$ ✓

Uniqueness: Follows from existence, since f and π are quotients (cancellation laws). ✓

The co-universal property of quotient groups

Theorem

Let $N \trianglelefteq G$ and $f\colon G \to K$ be a homomorphism such that $N \leq \mathrm{Ker}(f)$. Then

1. f uniquely factors through $\pi\colon G \to G/N$ (i.e., $\exists! h\colon G/N \to K$ such that $g = h \circ \pi$).
2. h is injective iff $\mathrm{Ker}(f) = N$.

Proof (ii)

Assume WLOG that f is onto. We just found the unique h such that

$$\begin{tikzcd} G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\pi"'] & & K \\ & G/N \arrow[ur, dashed, two heads, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & f(x) \\ & xN \arrow[ur, dashed, mapsto, "h"'] & \end{tikzcd}$$

Let $H = \mathrm{Ker}(f)$, and note that

$$\mathrm{Ker}(h) = \{xN \mid f(x) = 1_K\} = \{xN \mid x \in H\} = H/N.$$

Note that h is injective iff $|\mathrm{Ker}(h)| = 1$, or equivalently, $H = N$. $\square$

Co-universal property of quotient groups ⇒ FHT

Corollary: Fundamental homomorphism theorem

If $f\colon G \to H$ is a homomorphism, then $G/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$.

Proof

Let $K = \operatorname{Im}(f)$ and $N = \operatorname{Ker}(f)$ with canonical quotient map $\pi\colon G \to G/N$.

By construction, $\operatorname{Ker}(f) = N = \operatorname{Ker}(\pi)$.

By the co-universal property of quotient maps, f factors through the quotient:

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, two heads, "\pi"'] & & \operatorname{Im}(f) \\
& G/N \arrow[ur, hook, two heads, dashed, "\iota"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & f(x) \\
& xN \arrow[ur, mapsto, dashed, "\iota"'] &
\end{tikzcd}$$

Since $\operatorname{Ker}(f) = N$, the map ι is injective by Part (ii) of the previous theorem.

Therefore, ι is an isomorphism. $\square$

Abstracting the (co)-universal property

To motivate where we're going, let's rephrase what we just did as

> "*G/N is the* *largest quotient that collapses N*, *in that any other homomorphism collapsing N* *factors through $\pi\colon G \to G/N$ uniquely.*"

$$
\begin{tikzcd}
G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\pi"'] & & K \\
& G/N \arrow[ur, dashed, two heads, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & f(x) \\
& xN \arrow[ur, dashed, mapsto, "h"'] &
\end{tikzcd}
$$

Compare this to what we know about the commutator subgroup G':

> "*G/G' is the* *largest abelian quotient* *of G, in that any other homomorphism to an abelian group* *factors through $\alpha\colon G \to G/G'$ uniquely.*"

$$
\begin{tikzcd}
G \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\alpha"'] & & A \\
& G/G' \arrow[ur, dashed, two heads, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\alpha"'] & & f(x) \\
& xG' \arrow[ur, dashed, mapsto, "h"'] &
\end{tikzcd}
$$

Abstracting the (co)-universal property

The **co-universal property** of quotients came with a distinguished (maximal)

- group G/N, and
- canonical map $\pi\colon G \to G/N$.

Definition

A **co-universal pair** (C, χ) for a group G w.r.t. a property consists of:

- a group C, with
- an incoming map $\chi\colon G \to C$,

such that every $f\colon G \to H$ with the same property factors through χ uniquely.

I.e., there is a unique homomorphism $h\colon C \to H$ between co-domains such that $f = h \circ \chi$.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\chi"'] & & H \\ & C \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Abstracting the (co)-universal property

Proposition

If G has a co-universal pair (C, χ) w.r.t. some property, C is unique up to isomorphism.

Proof

Let (C, χ) and (C', χ') be co-universal. Start with (C, χ), and take $H = C'$ and $f = \chi'$.

By definition, $\exists! h \colon C \to C'$ such that $\chi' = h \circ \chi$. Reverse the roles, and we get:

$$
\begin{tikzcd}
G \arrow[rr, "\chi'"] \arrow[dr, "\chi"'] & & C' \\
& C \arrow[ur, dashed, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rr, "\chi"] \arrow[dr, "\chi'"'] & & C \\
& C' \arrow[ur, dashed, "h'"'] &
\end{tikzcd}
$$

We can "stack" one diagrams on the other, and vice-versa:

$$
\begin{tikzcd}
G \arrow[rrr, "\chi"] \arrow[drr, "\chi'"] \arrow[ddr, "\chi"'] & & & C \\
& & C' \arrow[ur, dashed, "h'"] & \\
& C \arrow[ur, dashed, "h"] \arrow[uurr, dashed, bend right, "\mathrm{Id}_C"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rrr, "\chi'"] \arrow[drr, "\chi"] \arrow[ddr, "\chi'"'] & & & C' \\
& & C \arrow[ur, dashed, "h"] & \\
& C' \arrow[ur, dashed, "h'"] \arrow[uurr, dashed, bend right, "\mathrm{Id}_{C'}"'] & &
\end{tikzcd}
$$

By uniqueness, $h \circ h' = \mathrm{Id}_C$ (left), and $h' \circ h = \mathrm{Id}_{C'}$ (right). Thus, $C \cong C'$. $\square$

Universal pairs and universal constructions

Definition

A **universal pair** (U, υ) for G w.r.t. a property consists of a group U and map $\upsilon \colon U \to G$, such that every other $f \colon H \to G$ with the same property factors through υ uniquely.

That is, $\exists! g \colon H \to U$ between the *domains* such that $f = \upsilon \circ g$.

$$\begin{tikzcd}
H \arrow[rr, "f"] \arrow[dr, dashed, "g"'] & & G \\
& U \arrow[ur, "\upsilon"'] &
\end{tikzcd}$$

Proposition (HW)

If G has a universal pair (U, υ) w.r.t. some property, then U is unique up to isomorphism.

Soon, we'll *define* concepts by a (co-)universal property.

These are examples of universal constructions.

Motivation: direct product vs. direct sums

Open-ended question

What is the limit of $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$*, as* $n \to \infty$*?*

- Define $\mathbb{R}^\infty$ to be the space of all infinite sequences

$$\mathbb{R}^\infty := \prod_{i=1}^{\infty} \mathbb{R} := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R}\}.$$

This space contains "vectors" such as $(1, 1, 1, \dots)$. We'll call it the "**direct product**."

- Define $\mathbb{E}^\infty$ to be the space of all finite sums, like

$$\mathbf{e} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = \sum_{i=1}^{n} a_i\mathbf{e}_i, \qquad ||\mathbf{v}|| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

We'll call this the "**direct sum**".

$$\mathbb{E}^\infty := \bigoplus_{i=1}^{\infty} \mathbb{R}\mathbf{e}_i := \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3 \oplus \cdots = \left\{ \sum_{i=1}^{k} a_i\mathbf{e}_i \mid a_i \in \mathbb{R},\ k \geq 1 \right\}$$
$$\cong \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R},\ \text{all but finitely many } a_j \text{ are zero}\}.$$

Motivation: direct product vs. direct sums

Define the canonical quotient maps for each $i = 1, 2, \ldots$ as

$$\pi_i \colon \mathbb{R} \times \mathbb{R} \times \cdots \longrightarrow \mathbb{R}, \qquad \pi_i \colon (a_1, a_2, \ldots) \longmapsto a_i.$$

The direct product is the "*smallest P that projects onto each factor*."

Given any family $f_i \colon X \to \mathbb{R}$ of maps, each f_i factors through the projection $\pi_i \colon P \to \mathbb{R}$.

$$\begin{tikzcd}
 & X \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
\mathbb{R} & \mathbb{R}\times\mathbb{R}\times\cdots \arrow[l, two heads, "\pi_1"] \arrow[r, two heads, "\pi_2"'] & \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[dl, mapsto, "f_1"'] \arrow[d, mapsto, dashed, "f"] \arrow[dr, mapsto, "f_2"] & \\
a_1 & (a_1, a_2, \ldots) \arrow[l, mapsto, "\pi_1"] \arrow[r, mapsto, "\pi_2"'] & a_2
\end{tikzcd}$$

Let's see why this fails if we tried to use $\mathbb{E}^\infty$ for P:

$$\begin{tikzcd}
 & X = \mathbb{R}\times\mathbb{R}\times\cdots \arrow[dl, two heads, "f_1"'] \arrow[d, two heads, dashed, "f"] \arrow[dr, two heads, "f_2"] & \\
\mathbb{R} & \underbrace{\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \cdots}_{\text{``\textit{too small}''}} \arrow[l, two heads, "\pi_1"] \arrow[r, two heads, "\pi_2"'] & \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x = (a_1, a_2, \ldots) \arrow[dl, mapsto, "f_1"'] \arrow[d, mapsto, dashed, "\nexists f"] \arrow[dr, mapsto, "f_2"] & \\
a_1 & ??? \arrow[l, mapsto, "\pi_1"] \arrow[r, mapsto, "\pi_2"'] & a_2
\end{tikzcd}$$

Motivation: direct product vs. direct sums

Define the natural inclusion map for each $j = 1, 2, \ldots$ as

$$\iota_j \colon \mathbb{R}\mathbf{e}_j \hookrightarrow \bigoplus_{i=1}^{\infty} \mathbb{R}\mathbf{e}_i, \qquad \iota_j \colon a_j\mathbf{e}_j \longmapsto a_j\mathbf{e}_j$$

The direct sum is the "*smallest S that each factor embeds into*."

Given any family $f_j \colon \mathbb{R}\mathbf{e}_j \to X$ of maps, each ι_j factors through the embedding $\iota_j \colon \mathbb{R}\mathbf{e}_j \hookrightarrow S$.

$$\begin{tikzcd}
 & X & \\
\mathbb{R}\mathbf{e}_1 \arrow[ur, "f_1"] \arrow[r, hook, "\iota_1"'] & \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \cdots \arrow[u, dashed, "f"'] & \mathbb{R}\mathbf{e}_2 \arrow[ul, "f_2"'] \arrow[l, hook', "\iota_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x_j & \\
a_1\mathbf{e}_1 \arrow[ur, mapsto, "f_1"] \arrow[r, mapsto, "\iota_1"'] & a_j\mathbf{e}_j \arrow[u, dashed, mapsto, "f"'] & a_2\mathbf{e}_2 \arrow[ul, mapsto, "f_2"'] \arrow[l, mapsto, "\iota_2"]
\end{tikzcd}$$

Let's see why this fails if we try to use $\mathbb{R}^\infty$ for S:

$$\begin{tikzcd}
 & \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \cdots & \\
\mathbb{R}\mathbf{e}_1 \arrow[ur, hook, "\iota_1"] \arrow[r, hook, "\iota_1"'] & \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots}_{\text{"\textit{too big}"}} \arrow[u, dashed, hook, "f"'] & \mathbb{R}\mathbf{e}_2 \arrow[ul, hook', "\iota_2"'] \arrow[l, hook', "\iota_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
??? \\
(1, 1, \ldots) \arrow[u, dashed, hook, "\nexists f"']
\end{tikzcd}$$

Returning to groups

Let $\{G_\alpha \mid \alpha \in A\}$ be a nonempty family of groups. We will define their product and co-product via a universal construction.

$$
\begin{tikzcd}
H \arrow[rr, "f_\alpha"] \arrow[dr, dashed, "g"'] & \text{``product''} & G_\alpha \\
& P \arrow[ur, "\pi_\alpha"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, hook, "\iota_\alpha"'] & \text{``co-product''} & H \\
& S \arrow[ur, dashed, "h"'] &
\end{tikzcd}
$$

Remark

Existence of the map needed to make these diagrams commute does *not* imply uniqueness from the cancellation laws—each is the "wrong type" of diagram for that.

The fact that there are such groups that guarantee uniqueness indicates that the definitions are capturing something fundamentally important.

Definition

The **product** of $\{G_\alpha \mid \alpha \in A\}$ is a group P with a family of homomorphisms $\{\pi_\alpha \colon P \to G_\alpha \mid \alpha \in A\}$, satisfying:

> *Given any group H and homomorphisms $f_\alpha \colon H \to G_\alpha$, there is a unique homomorphism $g \colon H \to P$ such that $\pi_\alpha \circ g = f_\alpha$ for all $\alpha \in A$.*

Co-products

Definition

The **co-product** of $\{G_\alpha \mid \alpha \in A\}$ is a group S with a family of homomorphisms $\{\iota_\alpha\colon G_\alpha \to S \mid \alpha \in A\}$, satisfying:

> *Given any group H and homomorphisms $f_\alpha\colon G_\alpha \to H$, there is a unique homomorphism $h\colon S \to H$ such that $h \circ \iota_\alpha = f_\alpha$ for all $\alpha \in A$.*

$$
\begin{tikzcd}
H \arrow[rr, "f_\alpha"] \arrow[dr, dashed, "g"'] & & G_\alpha \\
& P \arrow[ur, "\pi_\alpha"'] \arrow[u, phantom, "\text{``product''}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, hook, "\iota_\alpha"'] & & H \\
& S \arrow[ur, dashed, "h"'] \arrow[u, phantom, "\text{``co-product''}"] &
\end{tikzcd}
$$

Exercise (HW)

(i) If $\{G_\alpha \mid \alpha \in A\}$ has a product, it's unique up to isomorphism, and each π_α is surjective.

(ii) If $\{G_\alpha \mid \alpha \in A\}$ has a co-product, it's unique up to isomorphism, and each ι_α is injective.

Categories

Some constructions we've recently seen have analogues for other mathematical objects.

We can define the product and coproduct of sets, topological spaces, rings, vector spaces, etc.

Many structural results carry over, so we'd like to generalize these in a common framework.

The mathematical field that addresses these questions is called **category theory**.

Definition

A **category** $\mathcal{C}$ consists of

- a class $\mathrm{Ob}(\mathcal{C})$ of **objects**,
- a class $\mathrm{Hom}(\mathcal{C})$ of **morphisms** between objects, with identities, closure, and associativity.

- Examples of "objects" include sets, groups, rings, vector spaces, topological spaces, etc.,
- "Morphisms" are meant to be "structure-preserving maps."

Categories

Think of the category $\mathcal{C} = \textbf{Grp}$ of groups as a massive directed multigraph, where

- each node represents a group
- there is a directed edge from A to B for each homomorphism $f\colon A \to B$.

We require: identity, composition, and associativity.

$$\begin{tikzcd}
A \arrow[loop above, "\mathrm{Id}_A"] \arrow[r, bend left, "f"] & B \arrow[loop above, "\mathrm{Id}_B"] \arrow[l, bend left, "g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[loop above, "\mathrm{Id}_A"] \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[loop above, "\mathrm{Id}_A"] \arrow[r, "g"] & C \arrow[loop above, "\mathrm{Id}_C"]
\end{tikzcd}$$

Denote the morphisms from A to B by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

(i) Every group has an **identity morphism**: for every $A \in \mathrm{Ob}(\mathcal{C})$, there is $\mathrm{Id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ satisfying

$$f \circ \mathrm{Id}_A = f, \quad \text{for all } f \in \mathrm{Hom}_{\mathcal{C}}(A, B), \qquad \mathrm{Id}_A \circ g = g, \quad \text{for all } h \in \mathrm{Hom}_{\mathcal{C}}(B, A).$$

(ii) Morphisms are **closed under composition**:

$$\text{If } f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \mathrm{Hom}_{\mathcal{C}}(B, C), \text{ then } g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C).$$

(iii) Composition of morphisms is **associative**:

$$\text{If } f \in \mathrm{Hom}_{\mathcal{C}}(A, B), \ g \in \mathrm{Hom}_{\mathcal{C}}(B, C), \ h \in \mathrm{Hom}_{\mathcal{C}}(C, D), \text{ then } h \circ (g \circ f) = (h \circ g) \circ f.$$

Abstracting the notion of "one-to-one" and "onto"

Definition

Let $f, f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g, g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

1. g is a **monomorphism** if $g \circ f_1 = g \circ f_2$ implies $f_1 = f_2$.
2. f is an **epimorphism** if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$,

Sometimes, we'll say "*mono*" and "*epi*" (nouns) or *monic* and "*epic*" (adjectives).

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an **isomorphism** if it has a two-sided inverse.

That is, if $\exists g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

We say A and B are **equivalent**.

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

"monomorphism" (one-to-one)

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

"epimorphism" (onto)

$$\begin{tikzcd} A \arrow[loop above, "\mathrm{Id}_A = g \circ f"] \arrow[r, bend left, "f"] & B \arrow[loop above, "\mathrm{Id}_B = f \circ g"] \arrow[l, bend left, "g"] \end{tikzcd}$$

"equivalent" (isomorphic)

Abstracting the notion of "product" and "coproduct"

Definition

Consider a category $\mathcal{C}$ and a non-empty collection $\{B_i \mid i \in I\}$ of objects.

A **product** for $\{B_i\}$ is $P \in \mathrm{Ob}(\mathcal{C})$ with a family $\{\pi_i \in \mathrm{Hom}(P, B_i) \mid i \in I\}$ such that:

> *Given any $A \in \mathrm{Ob}(\mathcal{C})$ and $\left\{f_i \in \mathrm{Hom}_{\mathcal{C}}(A, B_i) \mid i \in I\right\}$, there is a unique $g \in \mathrm{Hom}_{\mathcal{C}}(A, P)$ such that $\pi_i \circ g = f_i$ for all $i \in I$.*

A **coproduct** for $\{B_i\}$ is $S \in \mathrm{Ob}(\mathcal{C})$ with $\{\iota_i \in \mathrm{Hom}(A_i, S) \mid i \in I\}$ such that:

> *Given any $B \in \mathrm{Ob}(\mathcal{C})$ and family $\left\{f_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B) \mid i \in I\right\}$, there is a unique $h \in \mathrm{Hom}_{\mathcal{C}}(S, B)$ such that $h \circ \iota_i = f_i$ for all $i \in I$.*

$$
\begin{tikzcd}
A \arrow[rr, "f_i"] \arrow[dr, dashed, "g"'] \arrow[drr, phantom, "\text{``product''}"] & & B_i \\
& P \arrow[ur, "\pi_i"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_i \arrow[rr, "f_i"] \arrow[dr, hook, "\iota_i"'] \arrow[drr, phantom, "\text{``co-product''}"] & & B \\
& S \arrow[ur, dashed, "h"'] &
\end{tikzcd}
\tag{1}
$$

It can be shown that the π_i's are epimorphisms, and ι_i's are monomorphisms.

A few counterintuitive facts

- *Isomorphisms $\neq$ mono + epi!* In the category **Rng**, the non-surjective morphism

$$g\colon \mathbb{Z} \longrightarrow \mathbb{Q}, \qquad g(n) = n$$

is both monic and epic.

$$R \overset{g_1}{\underset{g_2}{\rightrightarrows}} \mathbb{Z} \xrightarrow{f} \mathbb{Q} \qquad\qquad \mathbb{Z} \xrightarrow{g} \mathbb{Q} \overset{h_1}{\underset{h_2}{\rightrightarrows}} \mathbb{R}$$

The equality $f \circ g_1 = f \circ g_2$, implies $g_1 = g_2$, and $h_1 \circ g = h_2 \circ g$, forces $h_1 = h_2$.

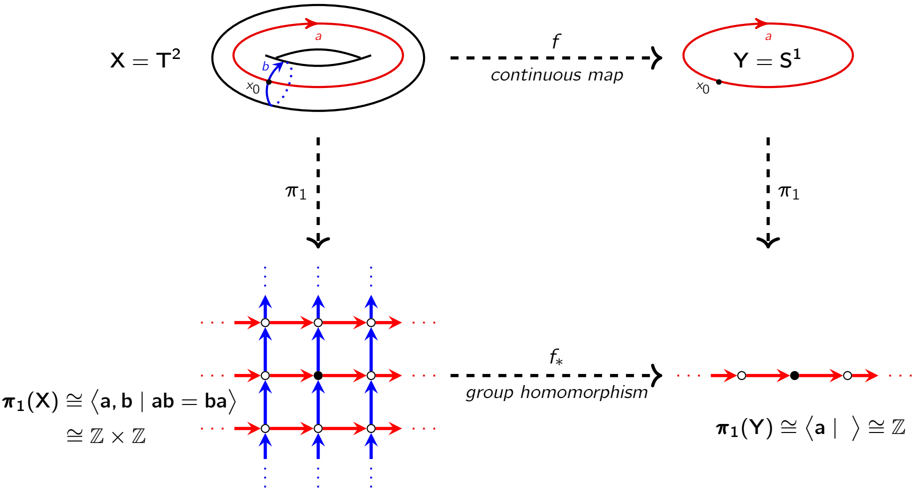
However, g is not an isomorphism because it does not have a left or a right inverse.

- *The same concept across different categories can seem very different!*

Category	Objects	Morphisms	Product	coproduct
Set	sets	functions	Cartesian product	disjoint union
Grp	groups	homomorphisms	direct product	free product
Ab	abelian groups	homomorphisms	direct product	direct sum
Ring	rings w/ 1	ring homomorphisms	direct product	free product
Field	fields	field embeddings	none	none
$\mathbf{Vect}_{\mathbb{F}}$	$\mathbb{F}$-vector spaces	linear functions	direct product	direct sum
Top	topological spaces	continuous maps	product topology	disjoint union

A functor from **Top** to **Grp**

Sometimes, there are structure-preserving maps between categories.



This is an example of a **functor**.

A functor from **Top** to **Grp**

The fundamental group of X is the group $\pi_1(X)$ of all "loops up to equivalence."

A continuous map $f\colon X \to Y$ induces a homomorphism

$$f_*\colon \pi_1(T^2) \longrightarrow \pi_1(S^1), \qquad f_*\colon (a,b) \longmapsto a.$$

Formally, π_1 is a functor from $\mathbf{Top}_\bullet$ to $\mathbf{Grp}$, defined as:

$$\begin{aligned} \pi_1\colon\ \mathrm{Ob}(\mathbf{Top}_\bullet) &\longrightarrow \mathrm{Ob}(\mathbf{Grp}) \\ X &\longmapsto \pi_1(X) \end{aligned} \qquad \begin{aligned} \mathcal{F}\colon\ \mathrm{Hom}(\mathbf{Top}_\bullet) &\longrightarrow \mathrm{Hom}(\mathbf{Grp}) \\ X \xrightarrow{f} Y \quad &\longmapsto \quad \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \end{aligned}$$

For arbitrary (pointed) topological spaces (X, x_0) and (Y, y_0):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\pi_1"] \\ \pi_1(X) \arrow[r, "f_*"] & \pi_1(Y) \end{tikzcd}$$

f is a contin. b/w topological spaces, with $f(x_0) = y_0$

f_* is a homomorphism b/w fundamental groups