2.1

equilibrium solutions are the solutions that can be "seen" in the direction field in Figure 8. They are shown plotted in blue in Figure 10.

Next we notice that $f(y) = 1 - y^2$ is positive if $-1 < y < 1$ and negative otherwise. Thus, if $y(t)$ is a solution to equation (1.24), and $-1 < y < 1$, then

$$y' = 1 - y^2 > 0.$$

Having a positive derivative, y is an increasing function.

How large can a solution $y(t)$ get? If it gets larger than 1, then $y' = 1 - y^2 < 0$, so $y(t)$ will be decreasing. We cannot complete this line of reasoning at this point, but in Section 2.9 we will develop the argument, and we will be able to conclude that if $y(0) = y_0 > 1$, then $y(t)$ is decreasing and $y(t) \to 1$ as $t \to \infty$.

On the other hand, if $y(0) = y_0$ satisfies $-1 < y_0 < 1$, then $y' = 1 - y^2 > 0$, so $y(t)$ will be increasing. We will again conclude that $y(t)$ increases and approaches 1 as $t \to \infty$. Thus any solution to the equation $y' = 1 - y^2$ with an initial value $y_0 > -1$ approaches 1 as $t \to \infty$.

Finally, if we consider a solution $y(t)$ with $y(0) = y_0 < -1$, then a similar analysis shows that $y'(t) = 1 - y^2 < 0$, so $y(t)$ is decreasing. As $y(t)$ decreases, its derivative $y'(t) = 1 - y^2$ gets more and more negative. Hence, $y(t)$ decreases faster and faster and must approach $-\infty$ as t increases. Typical solutions to equation (1.24) are shown in Figure 11. These solutions were found with a computer, but their qualitative nature can be found simply by looking at the equation.

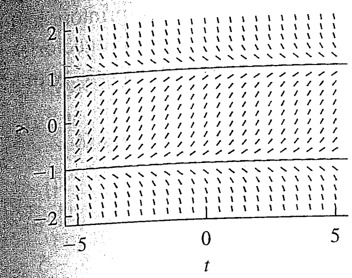

Figure 10. Equilibrium solutions to the equation $y' = 1 - y^2$.

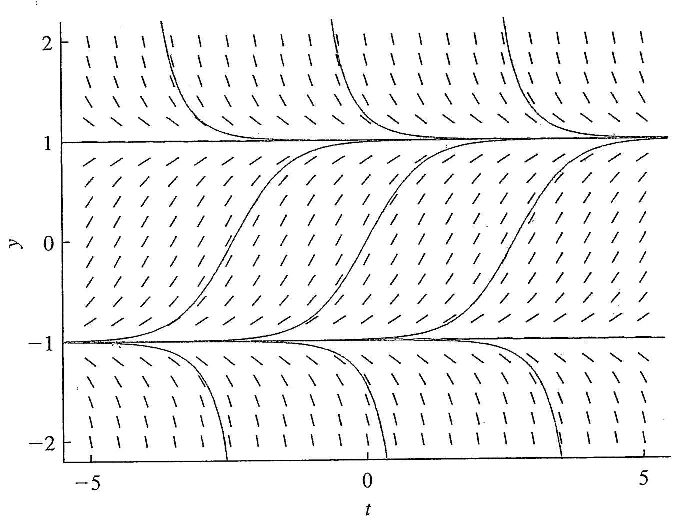

Figure 11. Typical solutions to the equation $y' = 1 - y^2$.

EXERCISES

In Exercises 1 and 2 , given the function ϕ, place the ordinary differential equation $\phi(t, y, y') = 0$ in normal form.

1. $\phi(x, y, z) = x^2 z + (1 + x)y$
2. $\phi(x, y, z) = xz - 2y - x^2$

In Exercises 3–6, show that the given solution is a general solution of the differential equation. Use a computer or calculator to sketch the solutions for the given values of the arbitrary constant. Experiment with different intervals for t until you have a plot that shows what you consider to be the most important behavior of the family.

3. $y' = -ty$, $y(t) = Ce^{-(1/2)t^2}$, $C = -3, -2, \dots, 3$
4. $y' + y = 2t$, $y(t) = 2t - 2 + Ce^{-t}$, $C = -3, -2, \dots, 3$
5. $y' + (1/2)y = 2\cos t$, $y(t) = (4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}$, $C = -5, -4, \dots, 5$
6. $y' = y(4 - y)$, $y(t) = 4/(1 + Ce^{-4t})$, $C = 1, 2, \dots, 5$

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7. A general solution may fail to produce all solutions of a differential equation. In Exercise 6, show that $y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

8. (a) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.
(b) Solve $t^2 + y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.
(c) Discuss the interval of existence for each of the solutions in part (b).
(d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

9. (a) Use implicit differentiation to show that $t^2 - 4y^2 = C^2$ implicitly defines solutions of the differential equation $t - 4yy' = 0$.
(b) Solve $t^2 - 4y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t - 4yy' = 0$.
(c) Discuss the interval of existence for each of the solutions in part (b).
(d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

10. Show that $y(t) = 3/(6t - 11)$ is a solution of $y' = -2y^2$, $y(2) = 3$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

11. Show that $y(t) = 4/(1 - 5e^{-4t})$ is a solution of the initial value problem $y' = y(4 - y)$, $y(0) = -1$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

In Exercises 12–15, use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solution's interval of existence.

12. $y' + 4y = \cos t$, $y(t) = (4/17)\cos t + (1/17)\sin t + Ce^{-4t}$, $y(0) = -1$

13. $ty' + y = t^2$, $y(t) = (1/3)t^2 + C/t$, $y(1) = 2$

14. $ty' + (t + 1)y = 2te^{-t}$, $y(t) = e^{-t}(t + C/t)$, $y(1) = 1/e$

15. $y' = y(2 + y)$, $y(t) = 2/(-1 + Ce^{-2t})$, $y(0) = -3$

16. Maple, when asked for the solution of the initial value problem $y' = \sqrt{y}$, $y(0) = 1$, returns two solutions: $y(t) = (1/4)(t + 2)^2$ and $y(t) = (1/4)(t - 2)^2$. Present a thorough discussion of this response, including a check and a graph of each solution, interval of existence, and so on. *Hint:* Remember that $\sqrt{a^2} = |a|$.

In Exercises 17–20, plot the direction field for the differential equation by hand. Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \le t \le 2$ and $-1 \le y \le 1$.

17. $y' = y + t$

18. $y' = y^2 - t$

19. $y' = t\tan(y/2)$

20. $y' = (t^2 y)/(1 + y^2)$

In Exercises 21–24, use a computer to draw a direction field for the given first-order differential equation. Use the indicated bounds for your display window. Obtain a printout and use a pencil to draw a number of possible solution trajectories on the direction field. If possible, check your solutions with a computer.

21. $y' = -ty$, $R = \{(t, y) : -3 \le t \le 3, -5 \le y \le 5\}$

22. $y' = y^2 - t$, $R = \{(t, y) : -2 \le t \le 10, -4 \le y \le 4\}$

23. $y' = t - y + 1$, $R = \{(t, y) : -6 \le t \le 6, -6 \le y \le 6\}$

24. $y' = (y + t)/(y - t)$, $R = \{(t, y) : -5 \le t \le 5, -5 \le y \le 5\}$

For each of the initial value problems in Exercises 25–28 use a numerical solver to plot the solution curve over the indicated interval. Try different display windows by experimenting with the bounds on y. *Note:* Your solver might require that you first place the differential equation in normal form.

25. $y + y' = 2$, $y(0) = 0$, $t \in [-2, 10]$

26. $y' + ty = t^2$, $y(0) = 3$, $t \in [-4, 4]$

27. $y' - 3y = \sin t$, $y(0) = -3$, $t \in [-6\pi, \pi/4]$

28. $y' + (\cos t)y = \sin t$, $y(0) = 0$, $t \in [-10, 10]$

Some solvers allow the user to choose dependent and independent variables. For example, your solver may allow the equation $r' = -2sr + e^{-s}$, but other solvers will insist that you change variables so that the equation reads $y' = -2ty + e^{-t}$, or $y' = -2xy + e^{-x}$, should your solver require t or x as the independent variable. For each of the initial value problems in Exercises 29 and 30, use your solver to plot solution curves over the indicated interval.

29. $r' + xr = \cos(2x)$, $r(0) = -3$, $x \in [-4, 4]$

30. $T' + T = s$, $T(-3) = 0$, $s \in [-5, 5]$

In Exercises 31–34, plot solution curves for each of the initial conditions on one set of axes. Experiment with the different display windows until you find one that exhibits what you feel is all of the important behavior of your solutions. *Note:* Selecting a good display window is an art, a skill developed with experience. Don't become overly frustrated in these first attempts.

31. $y' = y(3 - y)$, $y(0) = -2, -1, 0, 1, 2, 3, 4, 5$

32. $x' - x^2 = t$, $x(0) = -2, 0, 2$, $x(2) = 0$, $x(4) = -3, 0, 3$, $x(6) = 0$

33. $y' = \sin(xy)$, $y(0) = 0.5, 1.0, 1.5, 2.0, 2.5$

34. $x' = -tx$, $x(0) = -3, -2, -1, 0, 1, 2, 3$

35. Bacteria in a petri dish is growing according to the equation

$$\frac{dP}{dt} = 0.44P,$$

where P is the mass of the accumulated bacteria (measured in milligrams) after t days. Suppose that the initial mass of the bacterial sample is 1.5 mg. Use a numerical solver to estimate the amount of bacteria after 10 days.

2.2

The integral on the left contains the expression $y'(t)\,dt$. This is inviting us to change the variable of integration to y, since when we do that, we use the equation $dy = y'(t)\,dt$. Making the change of variables leads to

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.37}$$

Notice the similarity between (2.36) and (2.37). Equation (2.36), which has no meaning by itself, acquires a precise meaning when both sides are integrated. Since this is precisely the next step that we take when solving separable equations, we can be sure that our method is valid.

We mention in closing that the objects in (2.36), $h(y)\,dy$ and $g(t)\,dt$, can be given meaning as formal objects that can be integrated. They are called ***differential forms***, and the special cases like dy and dt are called ***differentials***. The basic formula connecting differentials dy and dt when y is a function of t is

$$dy = y'(t)\,dt,$$

the change-of-variables formula in integration. These techniques will assume greater importance in Section 2.6, where we will deal with exact equations. The use of differential forms is very important in the study of the calculus of functions of several variables and especially in applications to geometry and to parts of physics.

EXERCISES

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$
2. $xy' = 2y$
3. $y' = e^{x-y}$
4. $y' = (1 + y^2)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = x/(y + 2)$
8. $y' = xy/(x - 1)$
9. $x^2y' = y \ln y - y'$
10. $xy' - y = 2x^2y$
11. $y^3y' = x + 2y'$
12. $y' = (2xy + 2x)/(x^2 - 1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x$, $y(1) = -2$
14. $y' = -2t(1 + y^2)/y$, $y(0) = 1$
15. $y' = (\sin x)/y$, $y(\pi/2) = 1$
16. $y' = e^{x+y}$, $y(0) = 0$
17. $y' = (1 + y^2)$, $y(0) = 1$
18. $y' = x/(1 + 2y)$, $y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y$, $y(0) = 1$, $y(0) = -1$
20. $y' = -x/y$, $y(0) = 2$, $y(0) = -2$
21. $y' = 2 - y$, $y(0) = 3$, $y(0) = 1$
22. $y' = (y^2 + 1)/y$, $y(1) = 2$

23. Suppose that a radioactive substance decays according to the model $N' = N_0e^{-\lambda t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \tag{2.38}$$

24. The half-life of ^{238}U is 4.47×10^7 yr.

(a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.

(b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.

25. Tritium, ^{3}H, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of ^{3}H decays to 80 mg in 4 hours. Determine the half-life of ^{3}H.

26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, ^{99}Mb. If 10 g of ^{99m}Tc are "harvested" from the Molybdenum, how much of this sample remains after 9 hours?

27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of ^{131}I, how much of the isotope will be left after 20 days?

2.2

28. A substance contains two Radon isotopes, ^{210}Rn [$t_{1/2}$ = 2.42 h] and ^{211}Rn [$t_{1/2}$ = 15 h]. At first, 20% of the decays come from ^{211}Rn. How long must one wait until 80% do so?

29. Suppose that a radioactive substance decays according to the model $N = N_0e^{-\lambda t}$.

(a) Show that after a period of $T_\lambda = 1/\lambda$, the material has decreased to e^{-1} of its original value. T_λ is called the ***time constant*** and it is defined by this property.

(b) A certain radioactive substance has a half-life of 12 hours. Compute the time constant for this substance.

(c) If there are originally 1000 mg of this radioactive substance present, plot the amount of substance remaining over four time periods T_λ.

In the laboratory, a more useful measurement is the decay rate R, usually measured in disintegrations per second, counts per minute, etc. Thus, the ***decay rate*** is defined as $R = -dN/dt$. Using the equation $dN/dt = -\lambda N$, it is easily seen that $R = \lambda N$. Furthermore, differentiating the solution $N = N_0e^{-\lambda t}$ with respect to t reveals that

$$R = R_0e^{-\lambda t}, \tag{2.39}$$

in which R_0 is the decay rate at $t = 0$. That is, because R and N are proportional, they both decrease with time according to the same exponential law. Use this idea to help solve Exercises 30–31.

30. Jim, working with a sample of ^{131}I in the lab, measures the decay rate at the end of each day.

Time (days)	Counts (counts/day)	Time (days)	Counts (counts/day)
1	938	6	587
2	822	7	536
3	753	8	494
4	738	9	455
5	647	10	429

Like any modern scientist, Jim wants to use all of the data instead of only two points to estimate the constants R_0 and λ in equation (2.39). He will use the technique of ***regression*** to do so. Use the first method in the following list that your technology makes available to you to estimate λ (and R_0 at the same time). Use this estimate to approximate the half-life of ^{131}I.

(a) Some modern calculators and the spreadsheet Excel can do an exponential regression to directly estimate R_0 and λ.

(b) Taking the natural logarithm of both sides of equation (2.39) produces the result

$$\ln R = -\lambda t + \ln R_0.$$

Now $\ln R$ is a linear function of t. Most calculators, numerical software such as MATLAB®, and computer algebra systems such as *Mathematica* and Maple will do a linear regression, enabling you to estimate $\ln R_0$ and λ (e.g., use the MATLAB® command `polyfit`).

(c) If all else fails, plotting the natural logarithm of the decay rates versus the time will produce a curve that is almost linear. Draw the straight line that in your estimation provides the best fit. The slope of this line provides an estimate of $-\lambda$.

31. A 1.0 g sample of Radium 226 is measured to have a decay rate of 3.7×10^{10} disintegrations/s. What is the half-life of ^{226}Ra in years? *Note:* A chemical constant, called Avogadro's number, says that there are 6.02×10^{23} atoms per mole, a common unit of measurement in chemistry. Furthermore, the atomic mass of ^{226}Ra is 226 g/mol.

32. Radiocarbon dating. Carbon 14 is produced naturally in the earth's atmosphere through the interaction of cosmic rays and Nitrogen 14. A neutron comes along and strikes a ^{14}N nucleus, knocking off a proton and creating a ^{14}C atom. This atom now has an affinity for oxygen and quickly oxidizes as a $^{14}CO_2$ molecule, which has many of the same chemical properties as regular CO_2. Through photosynthesis, the $^{14}CO_2$ molecules work their way into the plant system, and from there into the food chain. The ratio of ^{14}C to regular carbon in living things is the same as the ratio of these carbon atoms in the earth's atmosphere, which is fairly constant, being in a state of equilibrium. When a living being dies, it no longer ingests ^{14}C and the existing ^{14}C in the now defunct life form begins to decay. In 1949, Willard F. Libby and his associates at the University of Chicago measured the half-life of this decay at 5568 ± 30 years, which to this day is known as the ***Libby half-life***. We now know that the half-life is closer to 5730 years, called the ***Cambridge half-life***, but radiocarbon dating labs still use the Libby half-life for technical and historical reasons. Libby was awarded the Nobel prize in chemistry for his discovery.

(a) Carbon 14 dating is a useful dating tool for organisms that lived during a specific time period. Why is that? Estimate this period.

(b) Suppose that the ratio of ^{14}C to carbon in the charcoal on a cave wall is 0.617 times a similar ratio in living wood in the area. Use the Libby half-life to estimate the age of the charcoal.

33. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton's law of cooling to calculate the victim's time of death. *Note:* The "normal" temperature of a living human being is approximately 37°C.

34. Suppose a cold beer at 40°F is placed into a warm room at 70°F. Suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed into the room.

35. Referring to the previous problem, suppose a 50° bottle of beer is discovered on a kitchen counter in a 70° room. Ten minutes later, the bottle is 60°. If the refrigerator is kept