MTHSC 208 (Differential Equations)
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HW 15
Due Monday March 9th, 2009

(1) Find the general solution of $x^2y'' - xy' - 3y = 0$.

(2) Find the general solution of $x^2y'' - xy' + 5y = 0$.

(3) Find the general solution of $x^2y'' - 3xy' + 4y = 0$.

(4) Consider the following ODE: $y'' - 2xy' + 10y = 0$.
- (a) Assume the solution has the form $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Plug $y(x)$ back into the ODE and find the recurrence relation for a_{n+2} in terms of a_n. Write down the general solution of the ODE.
- (b) Explicitly write out the coefficients a_n for $n \le 9$, in terms of a_0 and a_1. See the pattern? Write down the formula for a_n in terms of a_0 and a_1.
- (c) Find a basis for the solution space of the ODE (functions $y_0(x)$ and $y_1(x)$ such that the general solution is $y(x) = C_0y_0(x) + C_1y_1(x)$).
- (d) Find a non-zero polynomial solution for this ODE. [*Hint*: Make a good choice for a_0 and a_1.]
- (e) Are there any other polynomial solutions, excluding scalar multiples of the one you found in (d)? Why or why not?

(5) To solve the differential equation $y'' + x^2y'' - 4xy' + 6y = 0$, assume that the solution has the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
- (a) Plug $y(x)$ back into the ODE and determine the recurrence relation for the coefficients of the power series.
- (b) Explictly compute a_n in terms of a_0 and a_1 for $n \le 5$.
- (c) Write down a basis for the solution space.
- (d) How many distinct *polynomial* solutions are there to this ODE (i.e., up to scalar multiples)? Briefly justify your answer.

(6) The differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0,$$

where p is a constant, is known as *Legendre's equation*. We will solve it by first assuming that the solution is a power series centered at 0.
- (a) Find the recursion formula for a_{n+2} in terms of a_n.
- (b) Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \le n \le 9$.
- (c) Find a nonzero polynomial solution to this ODE, in the case where $p = 3$.
- (d) Find a basis for the space of solutions to the differential equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 12y = 0,$$

(7) We will find the general solution of *Airy's equation*: $y'' + xy = 0$.
- (a) Assume the solution is a power series. Find the recurrence relation of the coefficients of the power series. [*Hint*: When shifting the indices, one way is to let $m = n - 3$, then factor out x^{n+1} and find a_{n+3} in terms of a_n. Alternatively, you can find a_{n+2} in terms of a_{n-1}.]

(b) Show that $a_2 = 0$. [*Hint*: the two series for y'' and xy don't "start" at the same power of x, but for any solution, each term must be zero. (Why?)]

(c) Find the particular solution when $y(0) = 1$, $y'(0) = 0$, as well as the particular solution when $y(0) = 0$, $y'(0) = 1$.