MTHSC 208 (Differential Equations)
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HW 22
Due Wednesday April 15th, 2009

(1) Let X be a vector space over $\mathbb{C}$ (i.e., the contants are complex numbers, instead of just real numbers). If $\{v_1, v_2\}$ is a basis of X, then by definition, every vector v can be written uniquely as $v = C_1 v_1 + C_2 v_2$.

(a) Is the set $\{\frac{1}{2}v_1 + \frac{1}{2}v_2, \frac{1}{2}v_1 - \frac{1}{2}v_2\}$ a basis of X?

(b) Consider the ODE $y'' = 4y$. We know that the general solution is $y(t) = C_1 e^{2t} + C_2 e^{-2t}$, i.e., $\{e^{2t}, e^{-2t}\}$ is a basis for the solution space. Use (b), and the fact that $e^x = \cosh x + \sinh x$ to find a basis for the solution space involving hyperbolic sines and cosines, and write the general solution using these functions.

(2) We will find the function $u(x, t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \sin x + 3\sin 2x - 5\sin 7x.$$

(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.

(b) Using your solution to (a), find the general solution to the PDE

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}$$

subject to the *Dirichlet boundary conditions*:

$$u(0,t) = u(\pi,t) = 0\ .$$

(c) Solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = \sin x + 3\sin 2x - 5\sin 7x$.

(d) What is the steady-state solution, i.e., $\lim\limits_{t\to\infty} u(x,t)$?

(3) Find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x).$$

Note: The general solution will be exactly the same as in the previous problem. All you need to do again is Part (c) and (d) for this new initial condition, $u(x,0) = x(\pi - x)$. Additionally, sketch the bar and its initial heat distribution.

(4) We will find the function $u(x,t)$, defined for $0 \le x \le \pi$ and $t \ge 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u_x(0,t) = u_x(\pi,t) = 0, \qquad u(x,0) = 4 + 3\cos x + 8\cos 2x\ .$$

(a) Assume that $u(x,t) = f(x)g(t)$. Plug this back into the PDE and separate variables to get the *eigenvalue problem* (set equal to a constant λ). Solve for $g(t)$, $f(x)$, and λ.

(b) Using your solution to (a), find the general solution to the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the *Neumann boundary conditions*:

$$u_x(0,t) = u_x(\pi,t) = 0\ .$$

(c) Finally, solve the initial value problem, i.e., find the particular solution $u(x,t)$ that satisfies $u(x,0) = 4 + 3\cos x + 8\cos 2x$.
(d) What is the steady-state solution?

(5) Let $u(x,t)$ be the temperature of a bar of length 10, that is insulated so that no heat can enter or leave. Suppose that initially, the temperature increases linearly from 70° at one endpoint, to 80° at the other endpoint.
(a) Sketch the initial heat distribution on the bar, and express it as a function of x.
(b) Write down an initial value problem (a PDE with boundary and initial conditions) to which $u(x,t)$ is a solution (Let the constant from the heat equation be c^2).
(c) What will the steady-state solution be?

(6) Consider the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad u(x,0) = 3\sin\frac{5x}{2}.$$

(a) Describe a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.
(b) Assume that the solution is of the form $u(x,t) = f(x)g(t)$, and plug this into the PDE to get the eigenvalue problem (set equal to a constant λ). From this, write down two ODEs; one for f and one for g. Include boundary conditions for f.
(c) Solve the ODEs from the previous part for f and g. You may assume that $\lambda = -\omega^2$, (i.e., that $\lambda < 0$). Determine ω (be sure to show your work for this part!).
(d) Write down the general solution for $u(x,t)$, which solve the *mixed boundary conditions*:

$$u(0,t) = u_x(\pi,t) = 0\ .$$

(e) Find the particular solution for $u(x,t)$ satisfying the initial condition $u(x,0) = 3\sin(5x/2)$.
(f) What is the steady-state solution?