MTHSC 208 (Differential Equations)
Dr. Matthew Macauley
HW 24
Due Friday April 24th, 2009

(1) Which of the following functions are harmonic?
   (a) $f(x) = 10 - 3x$.
   (b) $f(x, y) = x^2 + y^2$.
   (c) $f(x, y) = x^2 - y^2$.
   (d) $f(x, y) = e^x \cos y$.
   (e) $f(x, y) = x^3 - 3xy^2$.

(2) (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = u(\pi, y) = 0,$$
$$u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x).$$

   (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$
$$u(x, 0) = u(x, \pi) = 0$$

   (c) By adding the solutions to parts (a) and (b) togeter, find the solution to the Dirichlet problem: Find $u(x, y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$
$$u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x).$$

   (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*

(3) Solve the following initial value problem for the heat equation in a square region: Find $u(x, y, t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$$
$$u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y.$$

You may assume that the nontrivial solutions to the eigenvalue problem

$$f_{xx} + f_{yy} = \lambda f, \qquad f(x, 0) = f(x, \pi) = f(0, y) = f(\pi, y) = 0,$$

are of the form

$$\lambda = -(m^2 + n^2), \qquad f(x, y) = b_{mn} \sin mx, sinny,$$

for $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$, where b_{mn} is a constant.

(4) Solve the following initial value problem for the heat equation in a square region: Find $u(x, y, t)$, where $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and $t \geq 0$ such that

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = (7 \sin x)\, y(\pi - y).
\end{aligned}$$

Sketch the initial heat distribution over this region. What is the steady-state solution?

(5) (a) Find the general solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$\begin{aligned}
&u(x, 0, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, \pi, t) = x(\pi - x).
\end{aligned}$$

(b) Sketch the steady-state solution.

(6) Solve the following initial value problem for a vibrating square membrane: Find $u(x, y, t)$, $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = p(x)q(y), \qquad \frac{\partial u}{\partial t}(x, y, 0) = 0.
\end{aligned}$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2, \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi, \end{cases} \qquad p(y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi. \end{cases}$$

Sketch the initial displacement of the square membrane. What is the long-term behavior of $u(x, y, t)$?