Q: What is a Differential Equation?
A: An equation involving a function and its derivatives.

Examples

- Finance: The rate of growth of an investment is proportional to the amount of the investment.

\[ P'(t) = r P(t) \] (often, just write \( P' = r P \)).

e.g., A mutual fund is increasing at a 10% rate.

\[ P'(t) = \frac{1}{10} P(t) \] (or \( P' = \frac{1}{10} P \)).

Note: We assume that interest is compounded continuously, i.e., at any point in time, the rate of change is \( \frac{1}{10} P \).

Big idea: Rate of change of a function is proportional to the function itself: \( f' = rf \).

large investment

large profit

small profit

small investment
**Biology**

A colony of bacteria grows at a rate proportional to its size.

\[ P'(t) = k \cdot P(t) \quad \text{Note: } r > 0 \text{ (why?)} \]

Note: It can't keep doubling forever. This is just a model, good for small \( t \).

e.g., 2 cells, 4 cells, 8 cells, 16 cells, etc.

---

**Chemistry**

A radioactive substance decays at a rate proportional to how much is remaining.

\[ y'(t) = k \cdot y(t) \quad \text{Note: } k < 0 \text{ (why?)} \]

Sample question: If there are 30 grams initially, and 20 grams after one year, what is its half-life?

Think: Is it even clear that "half-life" is well-defined?

- Compare this to investments.
The temperature of a cup of coffee cools at a rate proportional to: 
\[(\text{temp of coffee}) - (\text{ambient temp})].\]

Think: Imagine putting a cup of 75° water, and a cup of 200° water in a 72° room.

\[T'(t) = k(72 - T(t)), \quad k > 0\]

This is \underline{decay} towards a \underline{limiting value}.

What else exhibits this behavior in nature? (approximately)

- Earth's population
- Velocity of an object w/ air resistance.
  (here, "terminal velocity" plays the role of "ambient temp.")

But notice! Population grows \underline{differently}:

- when population is \underline{small}, it grows \underline{exponentially}
- when population is \underline{large}, it "decays \rightarrow\ \underline{carrying capacity}"

How do we put these two together?

Ans: Logistic equation. \[P'(t) = r \left(1 - \frac{P(t)}{M}\right) P(t)\]
Recall integral calculus.

Q: What is the antiderivative of \( f(x) = 2x \)?
A: \( F(x) = x^2 + C \).

Graphically:

All of these have derivative \( f(x) = 2x \).

Q: The velocity of a car is \( v(t) = 2t \). How far from home is it after \( t \) hrs?
A: \( x(t) = t^2 + C \)

\( x(t) \) is initial distance from home, "initial condition".

ODEs ("ordinary differential equations").

An investment takes 5 years to double.

Q: How much do we have after 8 years?
A: We don't know until we specify how much we have initially.

Q: solve \( f'(x) = 2x \) \( f(0) = 5 \)
\( f(x) = x^2 + C \) \( f(0) = C \Rightarrow C = 5 \)
\( f(x) = x^2 + 5 \)

solve \( f'(x) = 2x \) \( f(3) = 10 \)
\( f(x) = x^2 + C \) \( f(3) = 9 + C = 10 \Rightarrow C = 1 \)
\( f(x) = x^2 + 1 \)

These are initial value problems!
what if we didn't know how to solve \( F'(t) = 2t \)

Can we visualize the solution?

\[
\begin{array}{c|c}
 t & f'(t) \\
 0 & 0 \\
 1 & 2 \\
 2 & 4 \\
\end{array}
\]

Sketch the solution where:
- \( F(0) = -2 \)
- \( F(3) = -3 \)

This is a **direction field**. It allows us to visualize all solutions!

**Note:** Specifying \( f(x) = 5 \) completely determines the solution.

So, even if we can't solve an ODE, we can still graph its solution.

---

**How to solve ODEs**

Like integration, sometimes there's a method, other times it's art.

**Example:** Find all solutions to \( y' = ky \) \( \Rightarrow \frac{dy}{dt} = kt \)

"Magic:" multiply both by \( dt \): 
\[
\int y \, dy = \int ky \, dt
\]

Divide by \( y \) & integrate:
\[
\ln y = kt + C
\]

Take exp. on both sides:
\[
y = e^{kt+C}
\]

\( \text{Let } C = e^c \)

\[
y(t) = Ce^{kt}
\]

Q: what is \( C \)?
A: \( y(0) \). "initial condition."
(16)

Let's plot these: \( y(t) = y_0 e^{\frac{1}{10} t} \). Exponential growth. \( k > 0 \)

Exponential decay \( k < 0 \)

\( (y' = -\frac{1}{10} y) \quad y(t) = y_0 e^{-\frac{1}{10} t} \)

Note: \( \lim_{t \to \infty} y(t) = 0 \)

Question: Can two of these "phase lines" ever cross?

Answer: No (why?)

What if \( k = 0 \) ? \( (y' = 0) \quad \frac{dy}{dt} = 0 \Rightarrow y(t) = C. \)
Decay towards a limiting value

\[ y' = k(72 - y) \]
\[ \frac{dy}{dt} = -k(y - 72) \]
\[ \int \frac{dy}{y - 72} = \int k \, dt \]
\[ \ln|y - 72| = kt + C \]
\[ y - 72 = e^{kt + C} \]
\[ y(t) = 72 + Ce^{kt} \]

**Question:** What is \( C \)?

**Ans:** \( y(0) = 72 + C \).

"Initial temp difference"

---

**Initial value problems**

1. Solving an ODE yields a whole infinite family of solutions.
2. Once we specify a point on the \( (t, y(t)) \) plane, we completely determine which solution we have.

**Ex:** Consider the above example. Suppose \( y(0) = 90 \)

or suppose \( y(5) = 20 \).
Example: House sold in 2003 for $179,500 and was on sale for $319,500 in 2008.

(a) What was the rate of appreciation in value per year?

\[ P'(t) = r P(t), \quad \text{so} \quad P(t) = P_0 e^{rt} \]

\[ P(0) = P_0 = 179,500 \quad P(5) = 179,500 e^{5r} = 319,500 \]

Solve for \( r \).

\[ e^{5r} = \frac{319,500}{179,500} \]

\[ 5r = \ln \left( \frac{319.5}{179.5} \right) \Rightarrow r = \frac{1}{5} \ln \left( \frac{319.5}{179.5} \right) \approx 11.5\% . \]

Now, suppose the market has been increasing at 9% per year. How much is the house worth?

\[ A: \quad r = \frac{9}{100} \Rightarrow \quad P(5) = 179,500 e^{\frac{9}{100} \cdot 5} = \$281,512 . \]

Example: You have 10 grams of a radioactive substance. 3 years later, you have 4 grams.

(a) What is its half-life?

(b) How long until only 1 gram remains?

Recall: Half-life = amount of time it takes to have half of what you started with.

(a) \( y(t) = 10 e^{rt} \). Half-life: when \( y(t) = \frac{5}{10} = 10 e^{rt} \)

we also know: \( y(3) = 4 = 10 e^{3r} \)

- \( 4 = 10 e^{3r} \Rightarrow e^{3r} = \frac{2}{5} \Rightarrow 3r = \ln \left( \frac{2}{5} \right) \Rightarrow r = \frac{1}{3} \ln \left( \frac{2}{5} \right) \)
- \( 5 = 10 e^{\frac{1}{3} \ln \left( \frac{2}{5} \right) t} \Rightarrow \frac{1}{2} = e^{\frac{1}{3} \ln \left( \frac{2}{5} \right) t} \Rightarrow \ln \frac{1}{2} = \frac{1}{3} \ln \frac{2}{5} t \Rightarrow t_{1/2} = \frac{3 \ln 2}{\ln \frac{2}{5}} \)
Formal definition

- The order of an ODE is the order of the highest derivative that occurs in the equation.

- Equations involving partial derivatives are partial differential equations (PDEs). E.g., \( \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \) (heat equation).

- A 1st order equation of the form \( y' = f(y, t) \) is in normal form.

- An n-th order equation of the form \( y^{(n)} = f(y^{(n-1)}, \ldots, y'', y', y, t) \) is in normal form.

- We can write a 1st order ODE as \( \phi(t, y, y') = 0 \).

A solution to this ODE is any function \( y(t) \) such that \( \phi(t, y, y'(t)) = 0 \).

Often, there is an infinite family of solutions. E.g., \( y' = ky \) has general solution \( y(t) = Ce^{kt} \).

If we add an initial condition (e.g., \( y(0) = 5 \)), we get the particular solution \( y(t) = 5e^{kt} \).

Together, \( y' = ky \), \( y(0) = 5 \) is an initial value problem.
Sometimes, a solution isn't defined everywhere.

Think back to power series: \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \) for \( |x| < 1 \).

We defined the interval of convergence: \((-1, 1)\).

Similarly, in ODEs, we need to sometimes define the interval of existence.

**Ex:** \( y' = y^2 \), \( y(0) = 1 \)

\[ \frac{dy}{dt} = y^2 \Rightarrow \int y^{-2} \, dy = \int dt \Rightarrow -\frac{1}{y} = t + C. \]

\( \Rightarrow y = \frac{-1}{t+C} \) is the general solution.

\( y(0) = \frac{-1}{0+C} = 0 \Rightarrow C = -1. \)

So \( y(t) = \frac{-1}{t-1} \) is the particular solution.

\((-\infty, 1)\) is the interval of existence.

*Note:* we don't include the "bottom" curve because it doesn't go through \((0, 1)\).

Why? By definition, the interval of existence is the largest interval on which there is a solution curve that solves the IVP \( y' = y^2, \ y(0) = 1. \)
Separation of variables

When we have an ODE of the form \( \frac{dy}{dx} = \frac{g(t)}{h(y)} \),
we can take the following steps:

- Separate the variables: \( h(y) \, dy = g(t) \, dt \)
- Integrate both sides.
- Solve for \( y(t) \) (if possible)

\[
\frac{dy}{dx} = (1 + y^2) e^x \\
\int \frac{dy}{1 + y^2} = \int e^x \, dx \Rightarrow \tan^{-1} y = e^x + C \\
\Rightarrow y(t) = \tan(e^x + C).
\]

How to solve word problems

ex: Phosphorus. Initially, we have 1000 mg

After 10 hrs, we have 615 mg

What is the half-life?

\( y(t) = y_0 e^{kt} \) \( y_0 = 1000, \ y(10) = 615 \)

Find \( t \) such that \( y(t) = \)

we know:\( y(0) = 1000 e^{0k} = 650 \)

\[
1000 e^{10k} = 650 \Rightarrow e^{10k} = 0.65 \Rightarrow 10k = \ln 0.65 \Rightarrow k = \frac{10}{\ln 0.65}
\]

\[
1000 e^{kt} = 500 \Rightarrow e^{kt} = \frac{1}{2} \Rightarrow kt = \ln \frac{1}{2} \Rightarrow t = \frac{1}{k} \ln \frac{1}{2}
\]

\[
\Rightarrow t = \frac{10 \ln 0.65}{\ln 0.65}
\]