MTHSC 208, WEEK 1

Q: What is a Differential Equation?

A: An equation involving a <u>function</u> and its <u>derivative(s)</u>.

<u>Examples</u>

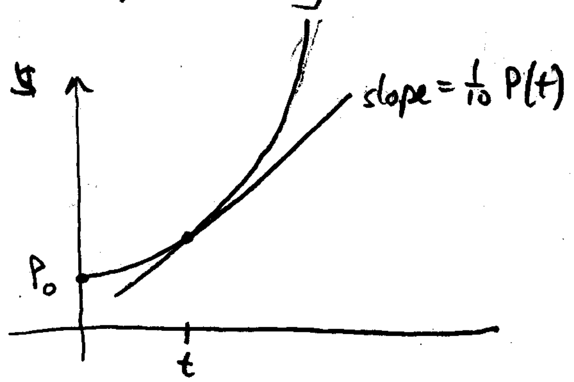
- $\boxed{\text{Finance}}$ The $\overbrace{\text{rate of growth of an investment}}^{P'(t)}$ is proportional to the $\underbrace{\text{amount of the investment}}_{P(t)}$.

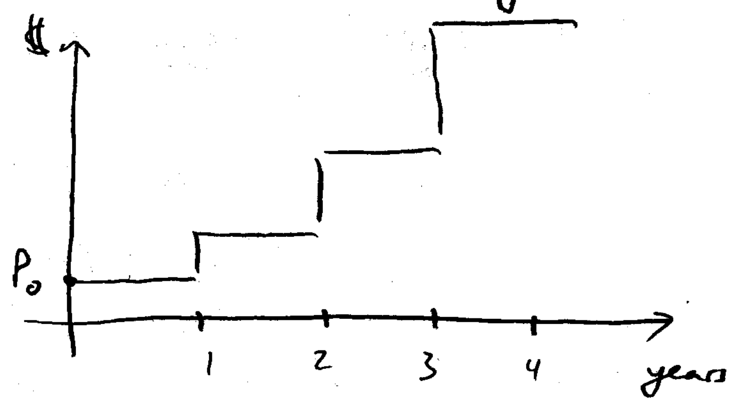
$P'(t) = r\,P(t)$ (often, just write $P' = rP$).

e.g., A mutual fund is increasing at a 10% rate.

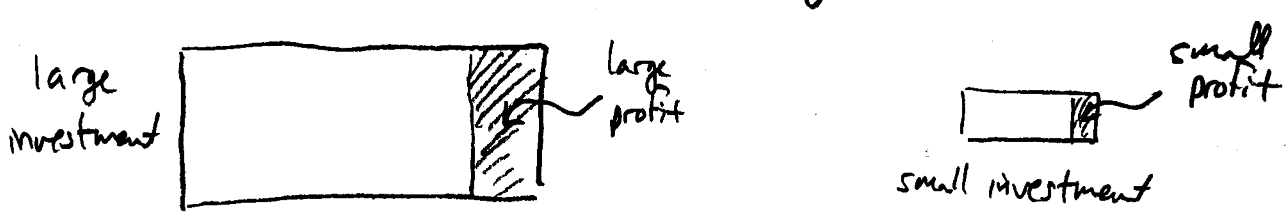
$P'(t) = \frac{1}{10} P(t)$ (or $P' = \frac{1}{10} P$).

<u>Note</u>: We assume that interest is compounded continuously, i.e, at any point in time, the rate of change is $\frac{1}{10} P$.





<u>Big idea</u>: Rate of change of a function is proportional to the function itself: $f' = rf$.



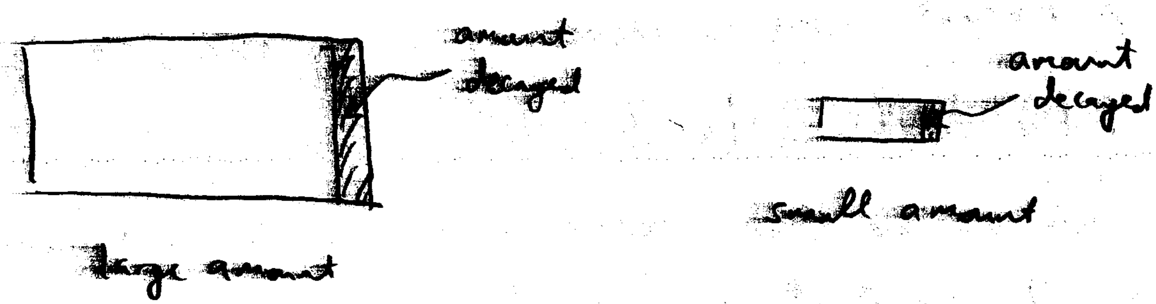
Biology: A colony of bacteria grows at a rate proportional to its size.

$$P'(t) = k\,P(t)$$ <u>Note:</u> $r > 0$ (why?)

<u>Note:</u> It can't keep doubling forever. This is just a model, good for small t.

e.g., 2 cells, 4 cells, 8 cells, 16 cells, etc.

Chemistry A radioactive substance decays at a rate proportional to how much is remaining.



$$y'(t) = k\,y(t)$$ <u>Note:</u> $k < 0$ (why?)

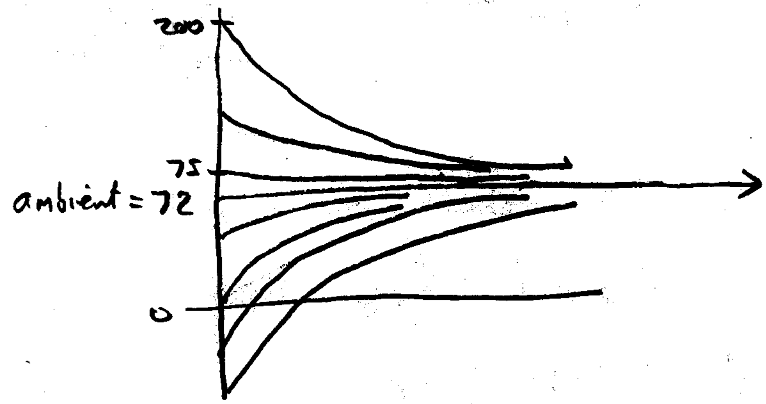
<u>Sample question:</u> If there are 30 grams initially, and 20 grams after one year, what is its half-life?

<u>Think:</u>
- Is it even clear that "half-life" is well-defined?
- Compare this to investments.

Physics The temperature of a cup of coffee cools at a rate proportional to: "(temp of coffee) − (ambient temp)".

<u>Think</u>: Imagine putting a cup of 75° water, and a cup of 200° water in a 72° room

$$T'(t) = k\,(72 - T(t)), \quad k > 0$$



This is <u>decay towards a limiting value.</u>

What else exhibits this behavior in nature? (approximately)

- Earth's population
- Velocity of an object w/ air resistance.
  (here, "terminal velocity" plays the role of "ambient temp.")

<u>But notice!</u> Population grows differently:

when population is <u>small</u>, it grows <u>exponentially</u>
when population is <u>large</u>, it "decays ⟶ carrying capacity."

How do we put these two together?

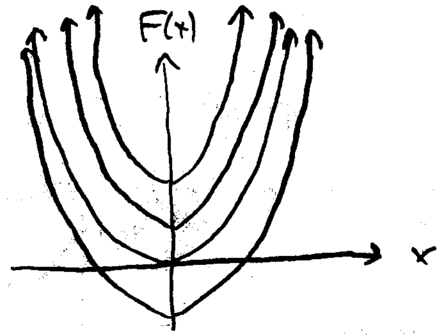
Ans: Logistic equation. $P'(t) = \underbrace{r\left(1 - \frac{P(t)}{M}\right)}_{\text{decay} \to M}\ \underbrace{P(t)}_{\text{exp. growth}}$

Recall integral calculus.

Q: what is the antiderivative of $f(x) = 2x$?

A: $F(x) = x^2 + C$.

Graphically:



All of these have derivative $f(x) = 2x$.

Q: The velocity of a car is $x'(t) = 2t$
How far from home is it after t hrs?

A: $x(t) = t^2 + C$

$\uparrow$ initial distance from home,
"initial condition"

ODEs ("ordinary differential equations").

An investment takes 5 years to double.

Q: How much do we have after 8 years?

A: We don't know until we specify how much we have initially.

Q: solve $f'(x) = 2x \quad f(0) = 5$

$f(x) = x^2 + C \quad f(0) = C \Rightarrow C = 5$

$f(x) = x^2 + 5$

solve $f'(x) = 2x \quad f(3) = 10$

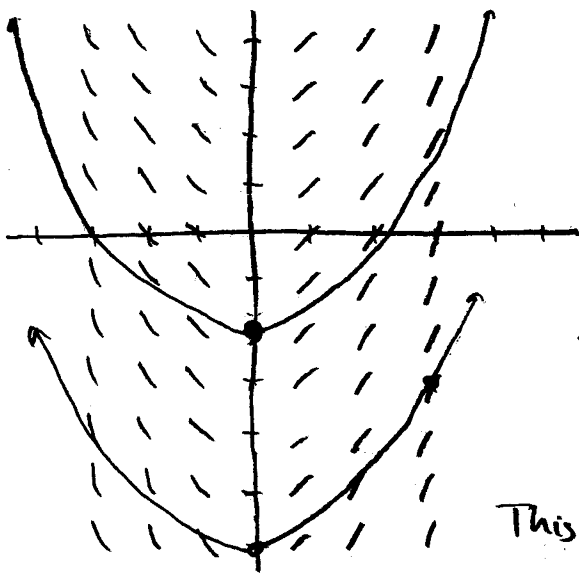
$f(x) = x^2 + C \quad f(3) = 9 + C = 10 \Rightarrow C = 1$

$f(x) = x^2 + 1$

These are initial value problems!

What if we didn't know how to solve $\underbrace{F'(t) = 2t}$.

Can we visualize the solution? ↑ slope of tangent line



$t=0 \quad f'(0) = 0$

$t=1 \quad f'(1) = 2$

$t=2 \quad f'(2) = 4$

Sketch the solution where: • $F(0) = -2$

• $F(3) = -3$

This is a direction field. It allows us to visualize all solutions!

Note: Specifying $f(a) = b$ completely determines the solution.

- So, even if we can't solve an ODE, we can still graph its solution.

How to solve ODEs

Like integration, sometimes there's a method, other times it's art.

Example: Find all solutions to $\boxed{y' = ky} \Rightarrow \frac{dy}{dt} = kt$

"Magic:" multiply thru by dt: $dy = ky\, dt$.

Divide by y & integrate: $\int \frac{1}{y}\, dy = \int k\, dt$

$$\ln y = kt + c$$

Take exp. on both sides: $y = e^{kt+c}$

$$= e^c e^{kt}$$

Let $C = e^c$. $\boxed{y(t) = Ce^{kt}}$

Q: what is C?

A: $y(0)$. "initial condition."

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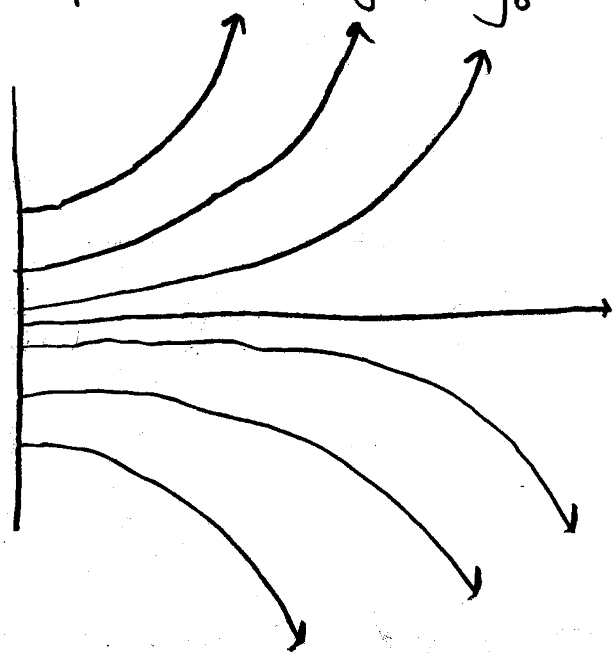
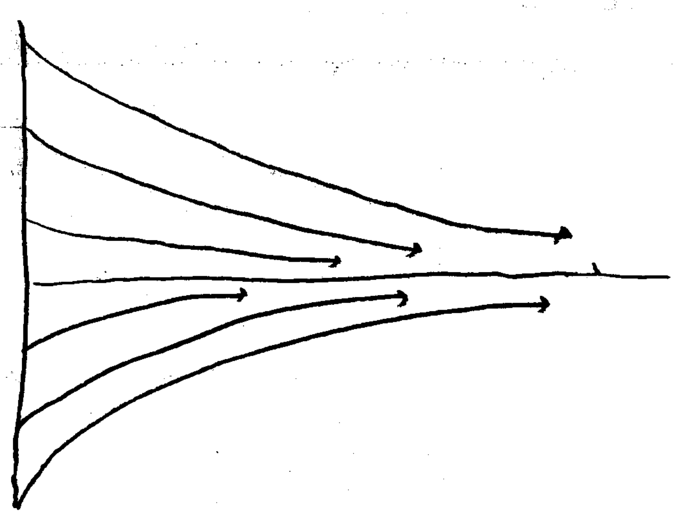
$(y' = \frac{1}{10} y)$

Let's plot these: $y(t) = y_0 e^{\frac{1}{10}t}$. Exponential growth. $\boxed{k>0}$



Exponential decay $\boxed{k<0}$ $(y' = -\frac{1}{10} y)$ $y(t) = y_0 e^{-\frac{1}{10}t}$

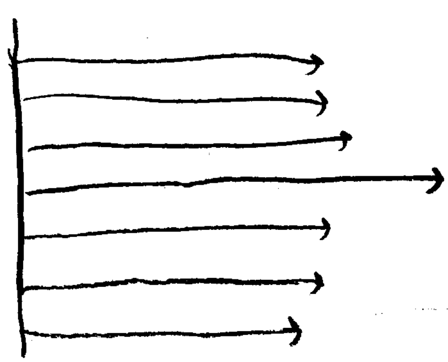


Note: $\lim_{t\to\infty} y(t) = 0$.

Question: Can two of these "phase lines" ever cross?

Ans: No (why?)

What if $\boxed{k=0}$? $(y'=0)$ $\frac{dy}{dt} = 0 \Rightarrow y(t) = C$.



Decay towards a limiting value

$$y' = k(72 - y)$$

$$\frac{dy}{dt} = -k(y-72)$$

$$\int \frac{dy}{y-72} = \int k \, dt$$

$$\ln|y-72| = kt + C$$

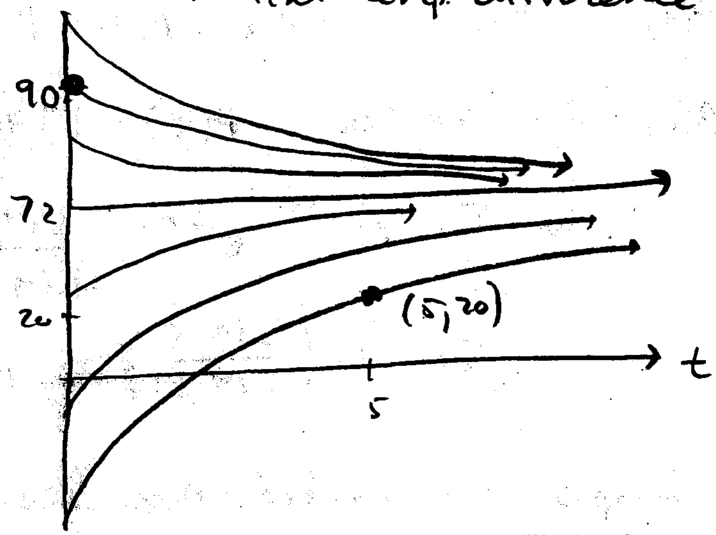
$$y - 72 = e^{kt+c}$$

$$\boxed{y(t) = 72 + Ce^{kt}}$$

Question: What is C?

Ans: $y(0) = 72 + C$.

"initial temp. difference"



Initial value problems

- Solving an ODE yields a whole infinite family of solutions.
- Once we specify a point on the $(t, y(t))$ plane, we <u>completely determine</u> which solution we have.

Ex: Consider the above example. Suppose $y(0) = 90$ or suppose $y(5) = 20$.

Example: House sells in 2003 for \$179,500 and was on sale for \$319,500 in 2008.

- what was the rate of appreciation in value per year?

$P'(t) = r\,P(t)$. sol'n $P(t) = P_0 e^{rt}$

$P(0) = P_0 = 179{,}500$ $\quad P(5) = 179{,}500\, e^{5r} = 319{,}500$

solve for r. $\quad e^{5r} = \dfrac{319{,}500}{179{,}500}$

$$5r = \ln\left(\frac{319.5}{179.5}\right) \Rightarrow r = \frac{1}{5}\ln\left(\frac{319.5}{179.5}\right) \approx 11.5\%.$$

Now, suppose the market has been increasing at 9%/year. How much is the house worth?

A: $r = \frac{9}{100}$, so $P(5) = 179{,}500\, e^{5\cdot\frac{9}{100}} = \$281{,}512.$

Example: You have 10 grams of a radioactive substance. 3 years later, you have 4 grams.

(a) What is its half-life?

(b) How long until only 1 gram remains?

Recall: Half-life = amt of time it takes to have half of what you started with.

(a) $y(t) = 10\,e^{rt}$. Half life: when $y(t) = \boxed{5 = 10e^{rt}}$

we also know: $y(3) = \boxed{4 = 10e^{3r}}$

- $4 = 10e^{3r} \Rightarrow e^{3r} = \frac{2}{5} \Rightarrow 3r = \ln\left(\frac{2}{5}\right) \Rightarrow r = \frac{1}{3}\ln\left(\frac{2}{5}\right)$
- $5 = 10\,e^{\frac{1}{3}\ln\left(\frac{2}{5}\right)t} \Rightarrow \frac{1}{2} = e^{\frac{1}{3}\ln\left(\frac{2}{5}\right)t} \Rightarrow \ln\frac{1}{2} = \frac{1}{3}\ln\frac{2}{5}\,t \Rightarrow \boxed{t_{1/2} = \frac{3\ln 1/2}{\ln 2/5}}$

Formal definitions

- The <u>order</u> of an ODE is the order of the highest derivative that occurs in the equation
- Equations involving partial derivatives are <u>partial differential equations</u> (PDEs). e.g., $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ (Heat equation).
- A 1^{st} order equation of the form $y' = f(y, t)$ is in <u>normal form</u>.
- An n^{th} order equation of the form $y^{(n)} = f(y^{(n-1)}, \dots, y'', y', y, t)$ is in <u>normal form</u>.
- We can write a 1^{st} order order ODE as $\phi(t, y, y') = 0$

  independent variable ↗ ↖ function

  A <u>solution</u> to this ODE is any function $y(t)$ such that $\phi(t, y, y'(t)) = 0$.

Often, there is an infinite family of solutions:

e.g., $y' = ky$ has <u>general solution</u> $y(t) = Ce^{kt}$.

If we add an <u>initial condition</u> (e.g., $y(0) = 5$), we get the <u>particular solution</u> $y(t) = 5e^{kt}$.

Together, $y' = ky$, $y(0) = 5$ is an <u>initial value problem</u>.

Sometimes, a solution isn't defined everywhere.

Think back to power series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$.

We defined the interval of convergence: $\underset{-1}{(}\!\!-\!\!\underset{0}{+}\!\!-\!\!\underset{1}{)}$

Similarly, in ODE's, we need to sometimes define the <u>interval of existence</u>.

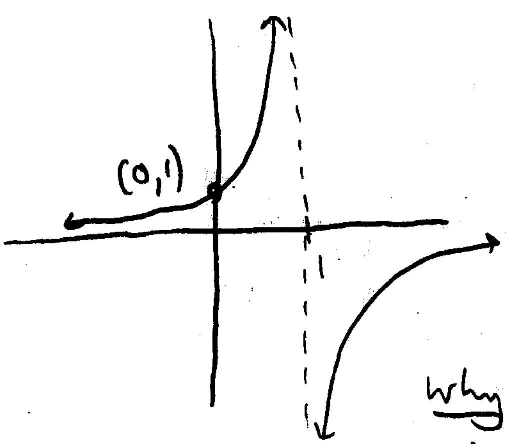
<u>ex</u>: $y' = y^2 \quad y(0) = 1$

$\frac{dy}{dt} = y^2 \Rightarrow \int y^{-2}\,dy = \int dt \Rightarrow -\frac{1}{y} = t + C.$

$\Rightarrow y = \frac{-1}{t+C}$ is the <u>general solution</u>.

$y(0) = \frac{-1}{0+C} = 0 \Rightarrow C = -1.$

So $y(t) = \frac{-1}{t-1}$ is the <u>particular solution</u>.



$(-\infty, 1)$ is the interval of existence.

<u>Note</u>: we don't include the "bottom" curve because it doesn't go through $(0,1)$.

<u>Why</u>? By definition: The interval of existence is the largest interval on which there is a <u>solution curve</u> that solves the IVP $y' = y^2$, $y(0) = 1$.

Separation of variables

When we have an ODE of the form $\frac{dy}{dt} = \frac{g(t)}{h(y)}$.

We can take the following steps:

- Separate the variables: $h(y)\,dy = g(t)\,dt$
- Integrate both sides.
- Solve for $y(t)$ (if possible)

ex: $\frac{dy}{dx} = (1+y^2)e^x$

$$\int \frac{dy}{1+y^2} = \int e^x\,dx \Rightarrow \tan^{-1} y = e^x + C$$

$$\Rightarrow y(t) = \tan(e^x + C)$$

How to solve word problems

ex: Phosphorus. Initially, we have 1000 mg

After 10 hrs, we have 615 mg

What is the half-life?

$y(t) = y_0 e^{kt}$ $\quad y_0 = 1000, \quad y(10) = 615$

Find t such that $y(t) = \boxed{1000e^{kt} = 500}$

we know: $y(10) = \boxed{1000e^{10k} = 650}$

$1000e^{10k} = 650 \Rightarrow e^{10k} = 0.65 \Rightarrow 10k = \ln 0.65 \Rightarrow \boxed{k = \frac{1}{10}\ln 0.65}$

$1000e^{kt} = 500 \Rightarrow e^{kt} = \frac{1}{2} \Rightarrow kt = \ln\frac{1}{2} \Rightarrow t = \frac{1}{k}\ln\frac{1}{2}$

$$\Rightarrow \boxed{t = \frac{10\ln 1/2}{\ln 0.65}}$$