MTHSC 208, WEEK 2

Big ideas from week 1: (Sections 2.1, 2.2)

- In many real-world situations, there are simple relations between a function and its derivatives. These can be expressed as ODEs.
- New technique for solving ODEs: Separation of variables.
- Exponential growth: $y' = ky \quad k>0$
  Exponential decay: $y' = ky \quad k<0$
  $\Big\}\; y(t) = y_0 e^{kt} \quad y(0) = y_0.$

  Decay $\longrightarrow$ value: $y' = k(A-y) \quad k>0 \quad y(t) = A + Ce^{kt} \quad y(0) = A + C.$
- Difference between the general solution & a particular solution (with initial conditions).

Recall:

Decay $\longrightarrow 0$ $\quad \frac{dy}{dt} = -ky \quad (k>0)$

$$\int \frac{dy}{y} = \int -k \, dt$$

$$\ln|y| = -kt + C$$

$$y(t) = e^{-kt+C}$$

$$\boxed{y(t) = Ce^{-kt}}$$

Decay $\longrightarrow A$ $\quad \frac{dy}{dt} = k(A-y) = -k(y-A)$

$$\int \frac{dy}{y-A} = \int -k \, dt$$

$$\ln|y-A| = -kt + C$$

$$y(t) - A = e^{-kt+C} = Ce^{-kt}$$

$$\boxed{y(t) = A + Ce^{-kt}}$$

Basic Physics

Falling object: $x(t)$ = height of an object

$v(t) = x'(t)$ = velocity

$a(t) = x''(t)$ = acceleration

Newton's 2^{nd} law: $F = ma$.

On Earth, $a = -g$, so $F = -mg$ ($g = 9.8\ m/s^2$ or $32\ ft/s^2$)

Think: Why the negative sign?

Together, we get $\cancel{m}x''(t) = -\cancel{m}g$

$$x''(t) = -g$$
$$x'(t) = -gt + C$$
$$x(t) = -\tfrac{1}{2}gt + Ct + D.$$

Note: $x'(0) = C$ initial velocity, v_0

$x(0) = D$ initial height, x_0.

Thus $\boxed{x(t) = -\frac{1}{2}gt + v_0 t + x_0}$

Now, assume there's air resistance.

Facts of air resistance:
1. No velocity, no air resistance
2. acts in opposite direction as velocity
3. Doesn't depend on x.

Together, we conclude $R(v) = -\underbrace{r(v)}_{\text{non-negative.}}\, v$

It's not exact, but a good model is $r(v) = \overset{\text{constant}}{r}\, v$.

Therefore, $F = -mg + R(v)$

$= -mg - rv.$

Newton's 2^{nd} law: $F = ma = m\frac{dv}{dt} = -mg - rv$

$$\boxed{\frac{dv}{dt} = -g - \frac{r}{m}v}$$

we can separate variables & solve...

OR, just remember that $\underbrace{v' = k(A - v)}_{\text{need to put in this form}}$ has sol'n $v(t) = A + Ce^{-kt}$

$$v' = -g - \frac{r}{m}v$$

$$= \frac{r}{m}\left(-\frac{mg}{r} - v\right) \qquad A = -\frac{mg}{r}, \quad k = -\frac{r}{m}$$

$$\boxed{v(t) = Ce^{-\frac{rt}{m}} - \frac{mg}{r}}$$

<u>Note</u>: $\lim_{t \to \infty} v(t) = -\frac{gm}{r}$ (terminal velocity)

Now, let's solve for $x(t)$.

<u>Recall</u>: $v(t) = \frac{dx}{dt}$

$$\frac{dx}{dt} = Ce^{-\frac{rt}{m}} - \frac{mg}{r}$$

Integrate: $x(t) = -\frac{m}{r}Ce^{-\frac{rt}{m}} - \frac{gm}{r}t + D$

const of integration

<u>Question</u>: Does a falling object ever reach terminal velocity?

<u>Example:</u> Drop a brick from the top of a building, 250m high. Mass = 2g, air resistance force $R(v) = -4v$.
How long will it take to reach the ground?

$$v' = -g - \frac{r}{m}v = -9.8 - \frac{4}{2}v = 2(-4.9 - v) = k(A - v)$$

$$k = 2, \quad A = -4.9 \Rightarrow v(t) = A + Ce^{-kt} = \boxed{-4.9 + Ce^{-2t}}$$

$v(0) = -4.9 + C = 0 \Rightarrow C = 4.9$. The particular solution is:

$$\boxed{v(t) = 4.9e^{-2t} - 4.9} = 4.9(e^{-2t} - 1).$$

We want time t, when $x(t) = 0$.

$$x(t) = 4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + D$$

$$x(0) = -\frac{4.9}{2} + D = 250 \Rightarrow D = 252.45$$

$$x(t) = 4.9\left(\tfrac{1}{2}e^{-2t} - t\right) + 252.45$$

The brick hits the ground when $x(t) = 0$.

$$4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + 252.45 = 0$$

Solve for t (with calculator / computer).

Linear equations

Recall high school algebra:

A linear equation is $f(x) = ax+b$.

In Math 208:

A (1st order) linear differential equation is $x' = a(t)\,x + f(t)$
(wrt x, $a(t)$ & $f(t)$ are constants)

A (1st order) homogeneous linear diff eq. is $x' = a(t)\,x$

Ex:

	$x' = t^2 x + 5$	linear
	$x' = t\,x^2 + 5$	non-linear
	$x' = t \sin x$	non-linear
	$x' = (\sin t)\,x$	linear, homogeneous
	$x' = t^3 - 2t^2 + t + 1$	linear
	$x' = x\,x'$	non-linear.

We'll see two methods for solving linear ODEs.

- Integrating factor
- Variation of parameters.

First, consider the homogeneous case: $x' = a(t)\,x$

$$\frac{dx}{dt} = a(t)\,x \Rightarrow \int \frac{dx}{x} = \int a(t)\,dt \Rightarrow \ln|x| = \int a(t)\,dt + C$$

$$|x| = e^{\int a(t)\,dt + C} \Rightarrow |x| = e^{C} e^{\int a(t)\,dt} \Rightarrow \boxed{x(t) = C e^{\int a(t)\,dt}}$$

Think: why did we drop the abs. value sign?

Now that we can solve $\underbrace{x'(t) = a(t)\,x(t)}_{\text{homogeneous eq'n}}$, let's solve $\underbrace{x'(t) = a(t)\,x(t) + f(t)}_{\text{inhomogeneous eq'n}}$.

Method #1: integrating factor ("product rule in reverse")

write as $\boxed{x' - ax = f}$.

Suppose "a" is a constant: $x' - ax$ "almost the product rule"

$$e^{-ax}x' - a e^{-ax}x \quad \text{(multiply by } e^{-ax}\text{)}$$
$$= (x e^{-ax})' \quad \text{(product rule)}$$

Let's try the same thing here.

$$e^{-\int a(t)\,dt}x' - a e^{-\int a(t)\,dt}x = f(t) e^{-\int a(t)\,dt}$$

$$\left(x e^{-\int a(t)\,dt}\right)' = f(t) e^{-\int a(t)\,dt} \quad \text{now, integrate both sides}$$

$$x(t) e^{-\int a(t)\,dt} = \int f(t) e^{-\int a(t)\,dt}\,dt \quad \text{solving for } x(t)\ldots$$

$$\boxed{x(t) = e^{\int a(t)\,dt} \int f(t) e^{-\int a(t)\,dt}\,dt}$$

Example 1: $y' = 2y + t$ Think: why won't separation of variables work?

$y' - 2y = t$. Integrating factor: e^{-2t}

$y'e^{-2t} - 2ye^{-2t} = te^{-2t}$

$\int (ye^{-2t})' = \int t e^{-2t}$

$$ye^{-2t} = -\tfrac{1}{4}e^{-2t}(2t+1) + C$$

$$\boxed{y(t) = -\tfrac{1}{2}t - \tfrac{1}{4} + Ce^{2t}}$$

Check: $y'(t) = -\frac{1}{2} + 2Ce^{2t}$, $-2y(t) = t + \frac{1}{2} + 2Ce^{2t}$

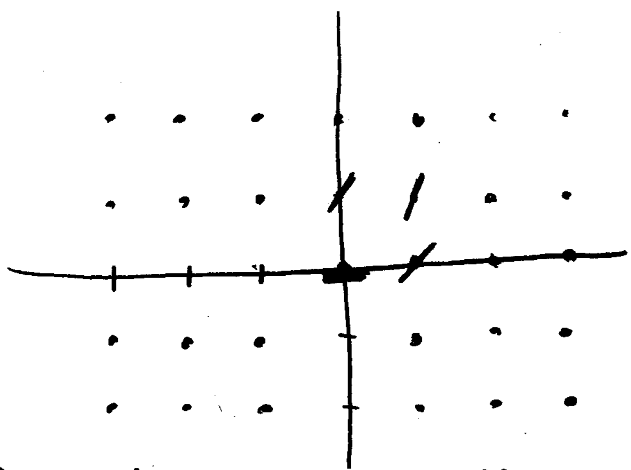
And $y' - 2y = t$ ✓ (i.e., $y(t)$ solves this ODE).

let's plot this: $y(t) = -\frac{1}{2}t - \frac{1}{4} + Ce^{2t}$

But how???

Go back to the ODE: $y' = 2y + t$.

<u>Method 1</u>: On a grid, draw the slope field point-by-point.



$y' = 2y + t$
$\uparrow$
slope

$(0,0) \rightsquigarrow y' = 0$

$(0,1)$: $y' = 2$

$(1,0)$: $y' = 1$

$(1,1)$: $y' = 3$

Downside: TEDIOUS!!!

<u>Method 2</u>: Isoclines.

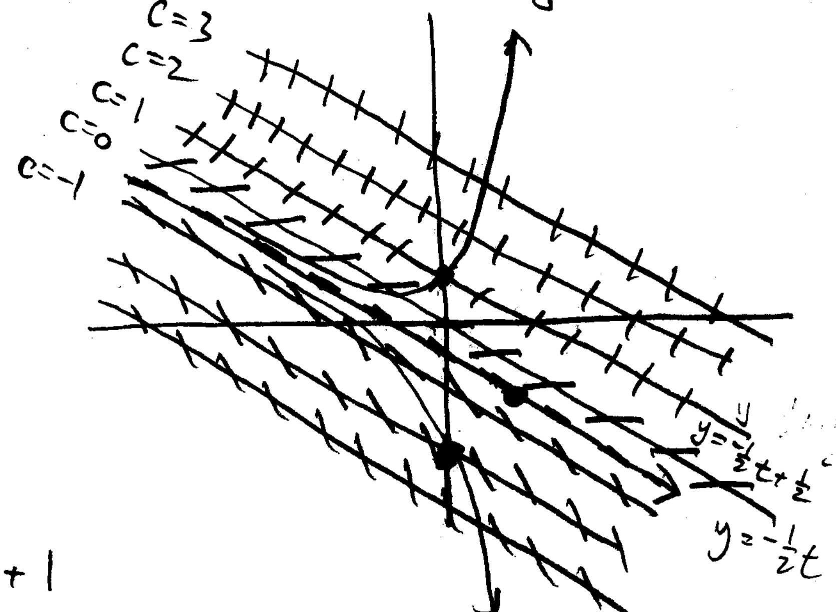
<u>Def</u>: An <u>isocline</u> is a line or curve on which $y' = \text{const}$.

<u>Q</u>: When is $y' = 0$?

<u>A</u>: When $2y + t = 0$,
i.e., $y = -\frac{1}{2}t$.

<u>Q</u>: When does $y' = 1$?

<u>A</u>: When $2y + t = 1$
i.e., $y = -\frac{1}{2}t + \frac{1}{2}$



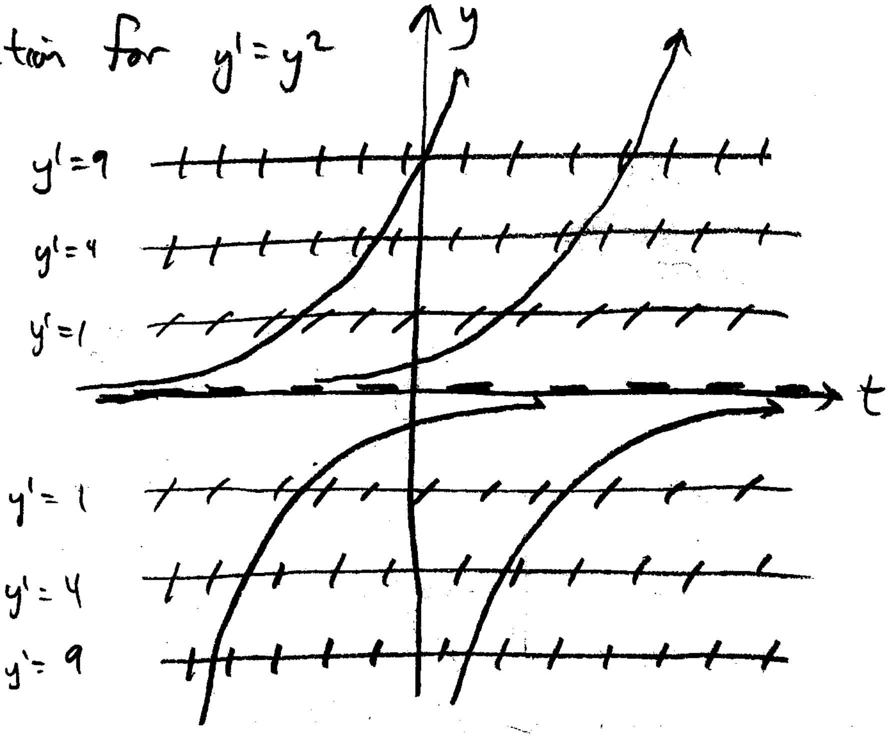
Continue: $y' = 2 \Rightarrow y = -\frac{1}{2}t + 1$

$y' = 3 \Rightarrow y = -\frac{1}{2}t + \frac{3}{2}$

$y = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}t - \frac{1}{4}$

Sketch the sol'n curves satisfying: (i) $y(0) = 1$ (ii) $y(1) = -\frac{3}{4}$ (iii) $y(0) = -1$

Example: Sketch the solution for $y' = y^2$



$y' = 0 \Rightarrow y = 0$

$y' = 1 \Rightarrow y = \pm 1$

$y' = 4 \Rightarrow y = \pm 2$

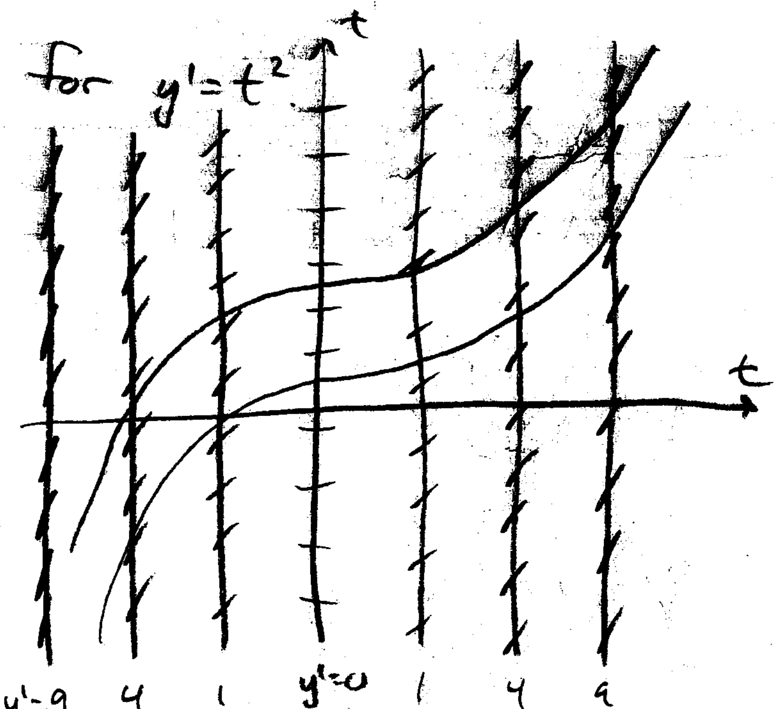
$y' = -1 \Rightarrow \varnothing$

Btw, what is $y(t)$?

$$\frac{dy}{dt} = y^2 \Rightarrow \int y^{-2}\,dy = \int dt$$

$$-\frac{1}{y} = t + c \Rightarrow \boxed{y(t) = \frac{-1}{t+c}}$$

Example: Sketch the solution for $y' = t^2$



$y(t) = \frac{1}{3}t^3 + c$

$y' = t^2$

$y' = 0 \Rightarrow t = 0$

$y' = 1 \Rightarrow t = \pm 1$

$y' = 4 \Rightarrow t = \pm 2$

$y' = 9 \Rightarrow t = \pm 3$

What have we done?

- We've started with an ODE & plotted it w/o even solving it!

Example: (of integrating factor AND isoclines).

$y' = y + e^{-t}$

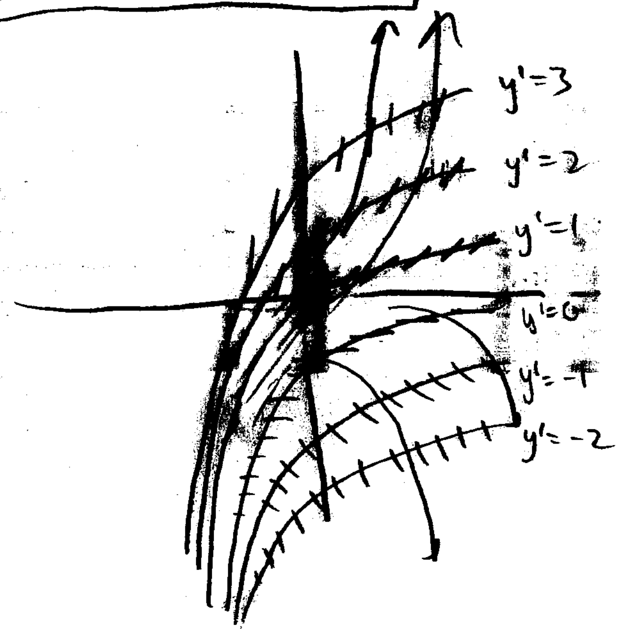
$y' - y = e^{-t}$ integrating factor: e^{-t}

$e^{-t}y' - e^{-t}y = e^{-2t}$

$\int (e^{-t}y)' = \int e^{-t}$

$e^{-t}y = -\frac{1}{2}e^{-2t} + c$ (Don't forget 'c'!)

$\boxed{y(t) = -\frac{1}{2}e^{-t} + Ce^{t}}$



Sketch the direction field.

(using isoclines)

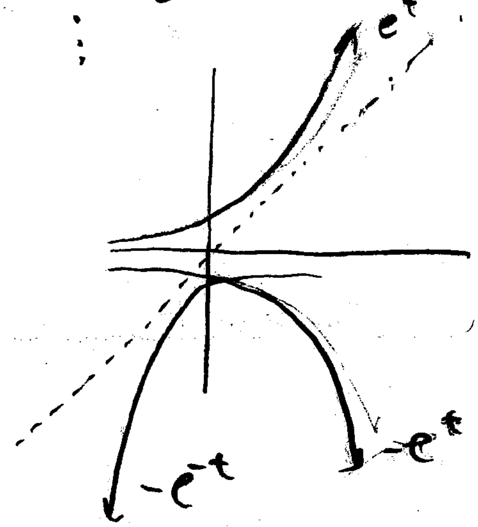
$y'=0$: $y = -e^{-t}$

$y'=1$: $1 = y + e^{-t}$
$y = 1 - e^{-t}$

$y'=2$ $y = 2 - e^{-t}$

$y'=3$ $y = 3 - e^{-t}$

$\vdots$



Recall the integrating factor method:

- $y' + 4y = t^2$ int. factor e^{4t}, $\frac{d}{dt}e^{4t} = 4e^{4t}$

$e^{4t}y' + 4e^{t}y = t^2e^{4t}$

$(e^{4t}y)' = t^2e^{4t}$ now solve...

$\boxed{16}$

- $y' + (\sin t)\, y = 1$ int. factor $e^{\cos t}$ $\frac{d}{dt} e^{\cos t} = \sin t\, e^{\cos t}$

$$e^{\cos t} y' + \sin t\, e^{\cos t} y = e^{\cos t}$$

$$(e^{\cos t} y)' = e^{\cos t}$$

- $y' - 12t^5 y = t^3$ int. factor e^{-2t^6} $\frac{d}{dt} e^{-2t^6} = -12t^5 e^{-2t^6}$

$$e^{-2t^6} y' - 12t^5 e^{-2t^6} y = e^{-2t^6} t^3$$

$$(e^{-2t^6} y)' = e^{-2t^6} t^3$$

- $y' + \frac{1}{t} y = 1$ int. factor $e^{\ln t} = t$

$$e^{\ln t} y' + \frac{1}{t} e^{\ln t} y = 1$$ $\frac{d}{dt} e^{\ln t} = \frac{1}{t} e^{\ln t} = 1$

$$t y' + y = 1$$

$$(t y)' = 1$$