MTHSC 208, WEEK 3

Week 2 summary: (Sections 2.2, 2.3, 2.4, supplemental)

- Falling objects, with & without air resistance. ($v' = -g - \frac{r}{m}v$)
- Linear ODEs: $x'(t) = a(t)\,x(t) + f(t)$

  Homogeneous if $f(t) = 0$.
- Integrating factor method for solving 1st order linear ODEs:

  $x' - ax = f$, int. factor $= e^{-\int a\,dt}$. "product rule in reverse"
- Sketching slope fields using isoclines (not in textbook!)

  Set $y' = \text{const.}$, plot the resulting line/curve.

Method #2 for solving linear 1st order ODEs: variation of parameters

Example: $y' = 2y + t$

Step 1: Solve the homogeneous part: $y_h' = 2y_h$

e.g., $y_h(t) = Ce^{2t}$.

Step 2: Assume the general sol'n is of the form $y(t) = v(t)\,y_h(t)$.

e.g., $y(t) = v(t)e^{2t}$.

Step 3: Plug this into the ODE & solve for $v(t)$.

e.g., $y' = 2y + t$

$$(ve^{2t})' = 2ve^{2t} + t$$

$$\cancel{2ve^{2t}} + v'e^{2t} = \cancel{2ve^{2t}} + t$$

$$v'e^{2t} = t$$

$$\int v'(t) = \int te^{-2t} \Rightarrow v(t) = -\tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} + C$$

<u>Step 4</u>: Plug back into $y(t) = v(t)\, y_h(t)$.

e.g., $y(t) = \left(-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c\right) e^{2t}$

$$\boxed{y(t) = -\frac{1}{2} t - \frac{1}{4} + C e^{2t}}$$

<u>Structure of the solution of a 1st order linear ODE:</u>

$$\boxed{y' = ay + f}$$

Solution has the form $\boxed{y(t) = v(t)\, y_h(t)}$

where $y_h(t) = e^{\int a(t)\, dt}$.

$$v'(t) = \frac{f(t)}{y_h(t)} = f(t)\, e^{-\int a(t)\, dt}$$

$$v(t) = \underbrace{\int f(t)\, e^{-\int a(t)\, dt}\, dt + C}_{\text{plug back into } y = v y_h}$$

$$(*) \quad y(t) = y_h(t) \int f(t)\, e^{-\int a(t)\, dt}\, dt + C y_h(t)$$

Let $y_p(t)$ be <u>any</u> particular sol'n to the ODE.

Then $y_p(t)$ is of the form in $(*)$ for some C, i.e.,

$$(**) \quad y_p(t) = y_h(t) \int f(t)\, e^{-\int a(t)\, dt}\, dt + C_p\, y_h(t)$$

Now, subtract $(*) - (**)$

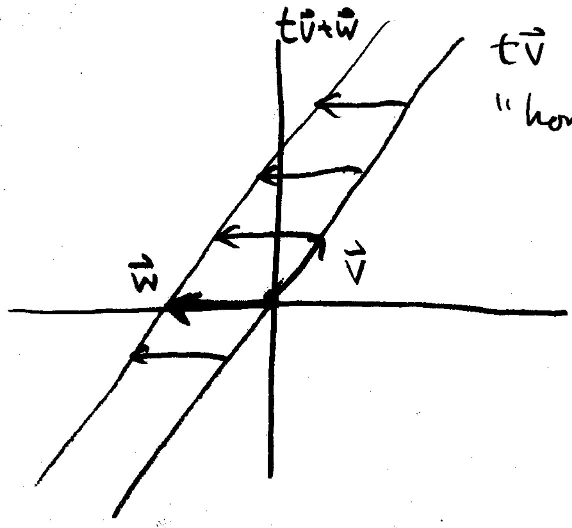
$$\underbrace{y(t) - y_p(t)}_{\text{this is a sol'n to homog. eqn.}} = (C - C_p)\, y_h(t).$$

Let $A = C - C_p$. Now, $y(t) - y_p(t) = A y_h(t)$, i.e., $\boxed{\boxed{y(t) = A y_h(t) + y_p(t)}}$

<u>Think</u>: What does this remind you of?
i.e, where have we seen this before?

$$y(t) = \underbrace{A}_{t}\ \underbrace{y_h(t)}_{\vec{v}} + \underbrace{y_p(t)}_{\vec{w}} \qquad A \in (-\infty, \infty) \qquad t \in (-\infty, \infty)$$

Recall vector calculus:

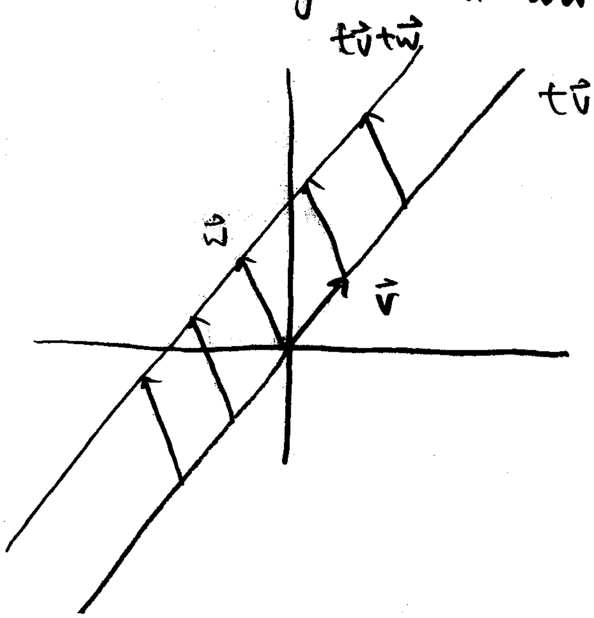


$t\vec{v}$ is this line thru $\vec{0}$ $(y = mx)$
"homogeneous"

<u>Q</u>: How do we parametrize a line <u>not</u> thru $\vec{0}$?

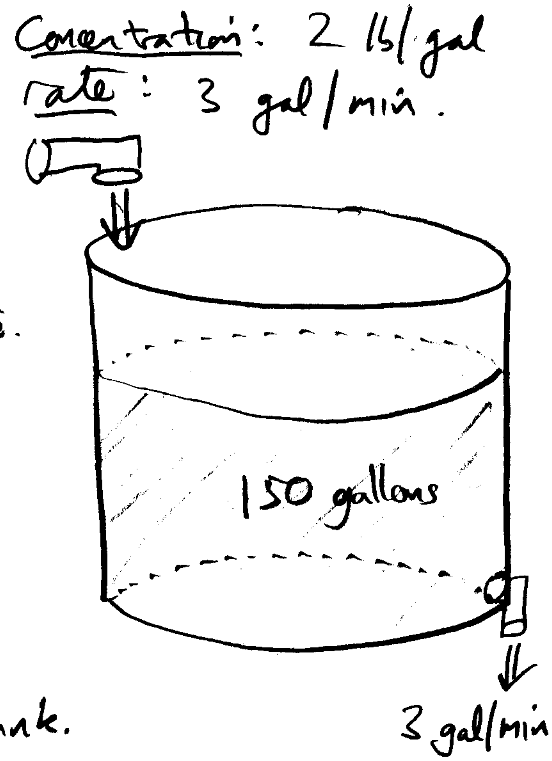
<u>A</u>: Add $\vec{w}$ to $t\vec{v}$, where $\vec{w}$ is <u>any</u> vector on the line.

<u>Q</u>: Why does "any" vector work?



<u>A</u>: Because any <u>particular</u> vector works! i.e, any (linear) line can be expressed as $A\vec{v}_h + \vec{v}_p$ $\quad A \in (-\infty, \infty)$
or $t\vec{v} + \vec{w}$

Mixing problems



Idea: Tank of fresh water

Salt water flows in at some rate.

Water drains at some rate

Q: What is the concentration of salt at some time t?

Let $X(t)$ = # pounds of salt in the tank.

i.e, $\frac{X(t)}{vol}$ = concentration of salt.

Big idea: "$\underbrace{\text{rate of change of salt}}_{X'(t)}$ = rate in − rate out"

rate in = (volume rate)(concentration)

$$= (3 \text{ gal/min})(2 \text{ lb/gal}) = 6 \text{ lb/min}$$

rate out = (volume rate)(concentration)

$$= (3 \text{ gal/min})\left(\frac{X(t) \text{ lb}}{150 \text{ gal}}\right) = \frac{1}{50} X(t) \text{ lb/min}$$

Putting this together: $\boxed{X'(t) = 6 - \frac{1}{50} X(t)}$

Note: This problem had:
- one tank
- rate in = rate out

But these need not hold!

Let's solve $x' + \frac{1}{50}x = 6$ integrating factor $e^{\frac{1}{50}t}$

OR separate variables

$$\frac{dx}{dt} = 6 - \frac{1}{50}x$$

$$\int \frac{dx}{6 - \frac{1}{50}x} = \int dt \Rightarrow -50 \ln\left(6 - \frac{1}{50}x\right) = t + C$$

$$\ln\left(6 - \frac{1}{50}x\right) = -50t + C$$

$$6 - \frac{1}{50}x = Ce^{-50t}$$

$$x(t) = 300 + Ce^{-50t}$$

Recall: Tank initially contains fresh water: $x(0) = 0$.

$$x(0) = 300 + C = 0 \Rightarrow C = -300$$

$$x(t) = 300 - 300e^{-50t}$$

Note: $\lim_{t\to\infty} x(t) = 300$

i.e., the amount of salt approaches 300 lbs.

Does this make sense?

Let's check:

As $t \to \infty$, the concentration approaches 2 lb/gal, and

vol = 150 gal ✓

Note: Mathematically, this is the same as:

- Heating / cooling
- Terminal velocity

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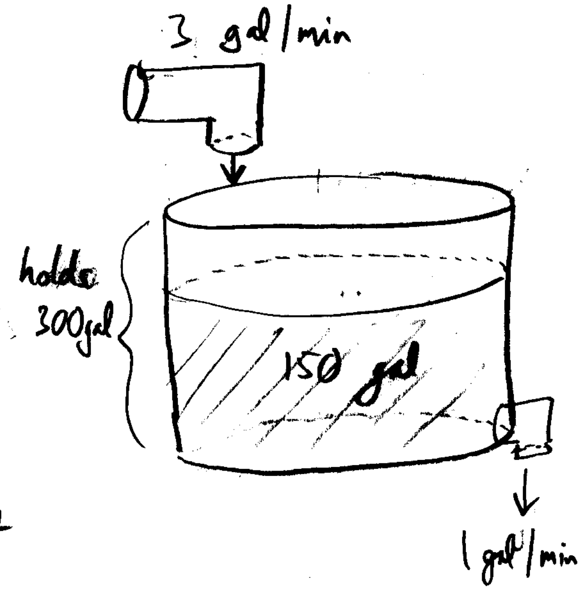
Now, consider a more complicated scenario.



Tank of fresh water.

Salt water flows in at a faster rate than it drains

Q: What is the concentration of salt at time t (before it overflows)?

And at what time does it overflow?

Again, let $x(t)$ = # lbs salt in the tank.

$$\text{Concentration} = \frac{x(t)}{\text{vol}}.$$

"Rate in" = (volume rate)(concentration)

$$= \left(3\ \frac{\text{gal}}{\text{min}}\right)\left(2\ \frac{\text{lbs}}{\text{gal}}\right) = 6\ \text{gal}/\text{min}.$$

"Rate out" = (volume rate)(concentration)

$$= \left(1\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{x(t)\ \text{lbs}}{150+2t}\right)$$

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\boxed{x'(t) = 6 - \frac{x(t)}{150+2t}}$$

Let's solve this: $x' + \frac{1}{150+2t}x = 6$ int factor: $e^{\frac{1}{2}\ln(150+2t)}$

$$\left(x e^{\frac{1}{2}\ln(150+2t)}\right)' = 6e^{\frac{1}{2}\ln(150+2t)}$$

$$x\sqrt{150+2t} = \int 6\sqrt{150+2t}\,dt$$

$$x(t)\,(150+2t)^{1/2} = 2(150+2t)^{3/2} + C$$

$$x(t) = 2(150+2t) + \frac{C}{\sqrt{150+2t}}$$

Initially, tank has only pure water.

$$x(0) = 0 = 300 + \frac{C}{\sqrt{150}} \Rightarrow C = -300\sqrt{150}$$

$$\boxed{x(t) = 300 + 4t - \frac{300\sqrt{150}}{\sqrt{150+2t}}}$$

Q: When does the tank overflow?

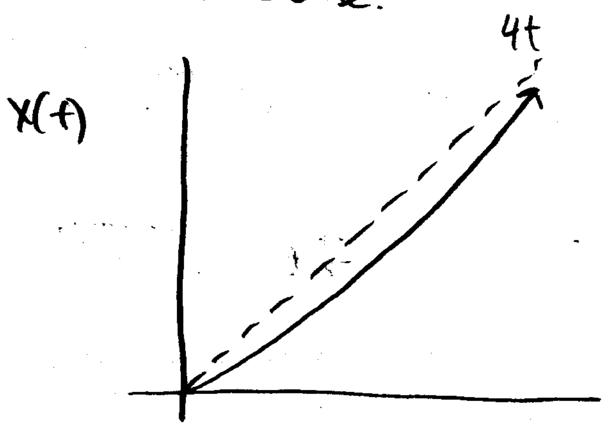
A: $\text{Vol}(t) = 150+2t = 300 \Longrightarrow \boxed{t = 75 \text{ min}}$

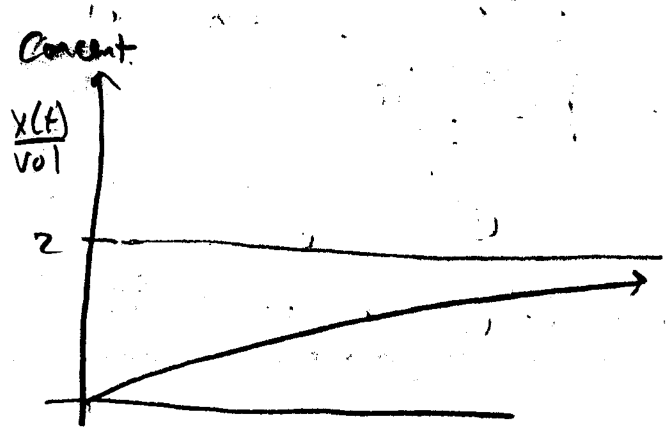
Q: What's the fixed amt of salt at this time.

A: $x(75) \approx 387.87$ lbs.

$$\text{Concent} = \frac{x(75)}{300\text{ gal}} \approx \frac{387.87}{300} \approx 1.29 \text{ lbs/gal}$$

This makes sense:





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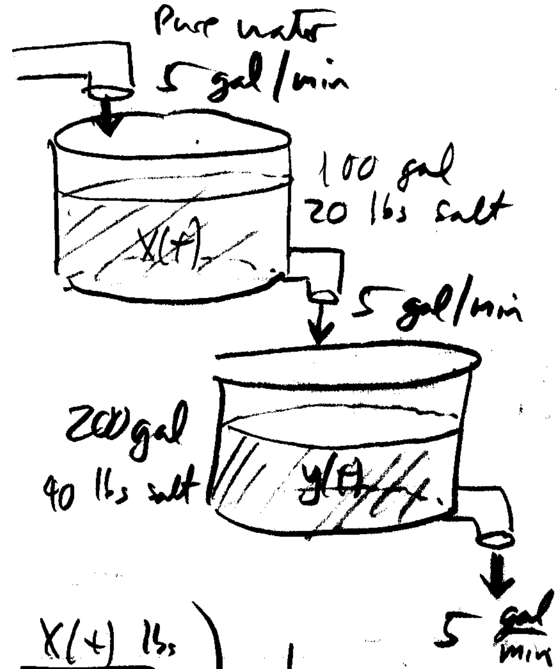
Mixing with 2 tanks:



let $x(t)$ = amt of salt in tank A

let $y(t)$ = amt of salt in tank B

Tank A: $x'(t) = (\text{rate in}) - (\text{rate out})$

rate in = (vol rate in)(concent.) = 0 lbs/min

rate out = (vol rate out)(concent.) $= \left(5 \frac{gal}{min}\right)\left(\frac{x(t)\ lbs}{100\ gal}\right) = \frac{1}{20}x$

Tank B: $y'(t) = (\text{rate in}) - (\text{rate out})$

rate in = (vol. rate)(concent.) $= \left(5 \frac{gal}{min}\right)\left(\frac{x(t)\ lbs}{100\ gal}\right) = \frac{1}{20}x$

rate out = (vol. rate)(concent.) $= \left(5 \frac{gal}{min}\right)\left(\frac{y(t)}{200\ gal}\right) = \frac{1}{40}y$

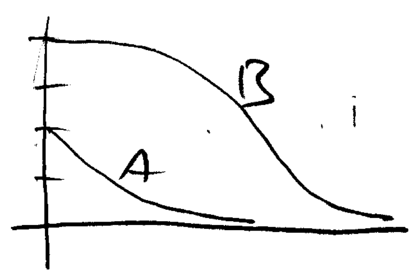
Together, we have:
$$\boxed{\begin{array}{ll} x' = -\frac{1}{20}x & x(0) = 20 \\ y' = \frac{1}{20}x - \frac{1}{40}y & y(0) = 40 \end{array}}$$

$x(t) = Ce^{-\frac{1}{20}t}$ $\quad x(0) = 20 \Rightarrow \boxed{x(t) = 20e^{-\frac{1}{20}t}}$

Plug into 2nd eqn: $y' = e^{-\frac{1}{20}t} - \frac{1}{40}y$

$y' + \frac{1}{40}y = e^{-\frac{1}{20}t}$. $\quad$ int. factor $e^{\frac{1}{40}t}$

$\left(ye^{\frac{1}{40}t}\right)' = e^{-\frac{1}{40}t} \Rightarrow ye^{\frac{1}{40}t} = -40e^{-\frac{1}{40}t} + C$

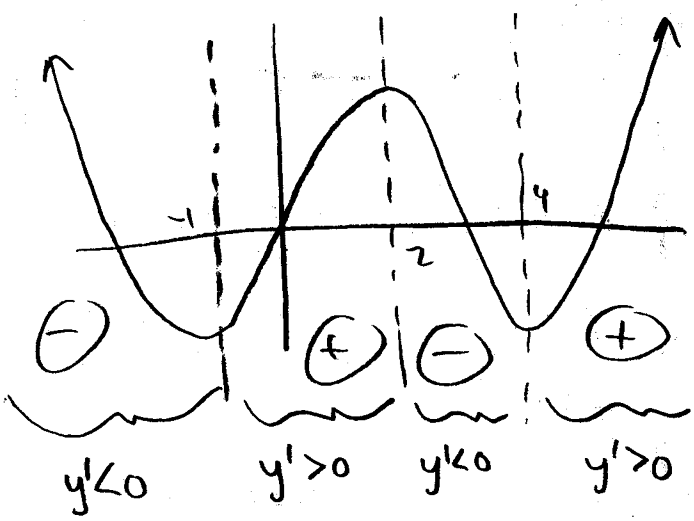


$y(t) = -40e^{-\frac{1}{20}t} + Ce^{-\frac{1}{40}t}$.

$y(0) = -40 + C = 40 \Rightarrow C = 80.$ $\quad \boxed{y(t) = -40e^{-\frac{1}{20}t} + 80e^{-\frac{1}{40}t}}$

Recall basic calculus

Suppose $y'(x) = (x+1)(x-2)(x-4)$

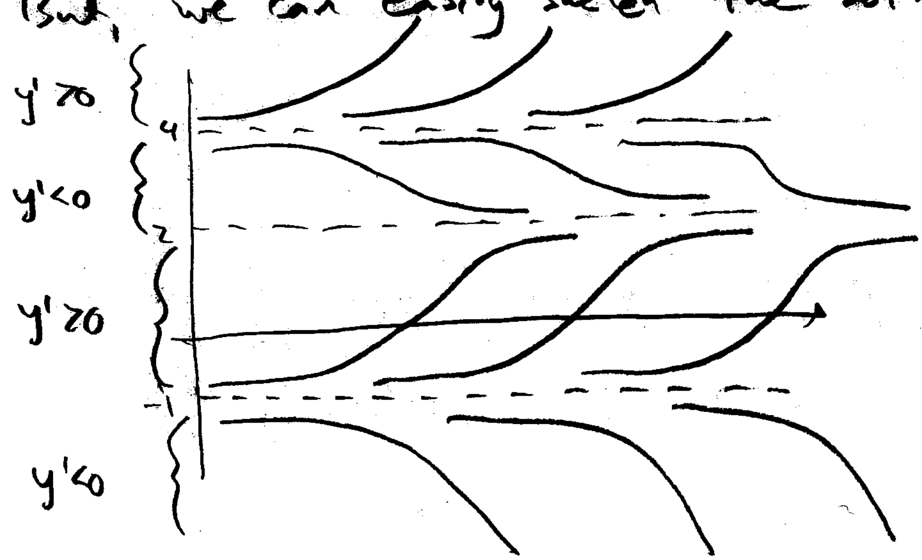


Now, in ODEs:

Suppose $y' = (y+1)(y-2)(y-4)$

How do we solve this? (No idea).

But, we can easily sketch the sol'n field.



* y' does not depend on t (it is autonomous).

This was even easier than using isoclines.