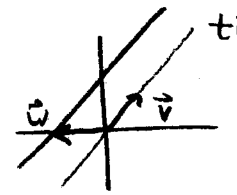
MTHSC 208, WEEK 4

Week 3 Summary:

- 3 different ways to solve 1^{st} order linear ODE^s: $y' - ay = F$
  - (i) Integrating factor: Multiply by $e^{-\int a(t)\,dt}$
  - (ii) Variation of parameters: Assume $y(t) = v(t)\, y_h(t)$
  - (iii) Homogeneous + particular sol'n: $y(t) = y_h(t) + y_p(t)$.
- Connection between general solution & parametrized lines: $t\vec{v} + \vec{w}$

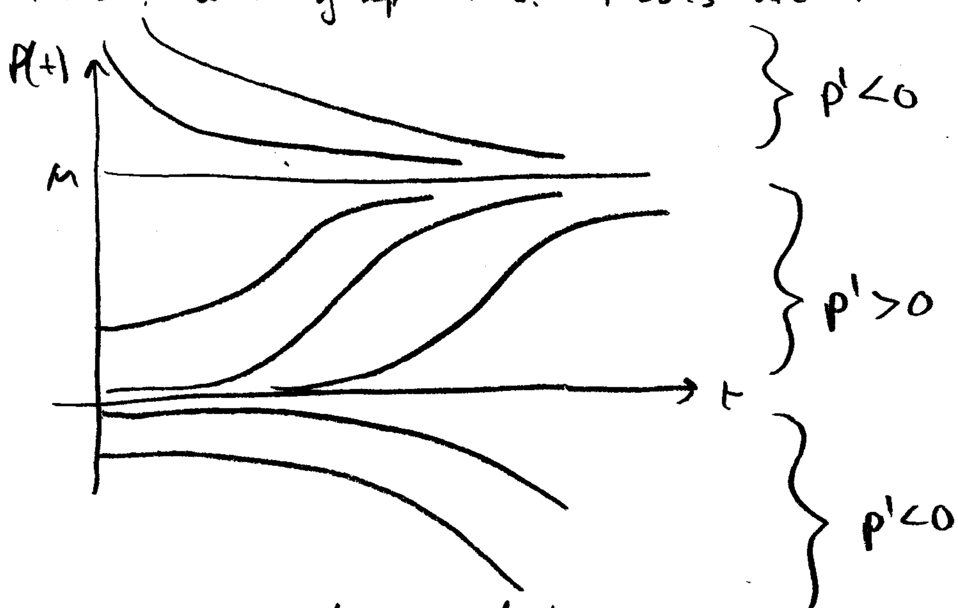
$$y(t) = C\, y_h(t) + y_p(t) \qquad\qquad c(t) = t\,\vec{v} + \vec{w}$$



- How to set up & solve mixing problems: $x'(t) = (\text{rate in}) - (\text{rate out})$
- How to plot autonomous ODEs quickly.

Logistic equation: $P' = rP\left(1 - \frac{P}{M}\right)$.

First, let's graph this: Roots are $P=0$ and $P=M$.



This models population growth

The constant M represents the carrying capacity.

- Suppose $P \approx 0$. Then $1-\frac{P}{M} \approx 1$.

  Therefore, $P' \approx rP$

  i.e, when there's plenty of space and resources, population grows exponentially.

- Now, suppose $P(t) \approx M$.

  $P' \approx \emptyset \Rightarrow P \approx rM(1-\frac{P}{M}) = r(M-P)$ decay $\to M$.

  i.e, for population $\approx M$, pop. size decays $\longrightarrow M$, like heating / cooling, terminal vel, etc.

<u>Solving the logistic equation:</u>

$$\frac{dP}{dt} = rP\left(1-\frac{P}{M}\right) = r\frac{1}{M}P(M-P)$$

$$\frac{M\,dP}{P(M-P)} = \int r\,dt = \int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP \quad \text{(partial fractions)}$$

$$rt + C = \ln|P| - \ln|M-P|$$

$$rt + C = \ln\left|\frac{P}{M-P}\right|$$

$$Ce^{rt} = \frac{P}{M-P} \Rightarrow MCe^{rt} - PCe^{rt} = P$$

$$MCe^{rt} = P(1-Ce^{rt})$$

$$P(t) = \frac{MCe^{rt}}{1+Ce^{rt}} \cdot \frac{\frac{1}{C}e^{-rt}}{\frac{1}{C}e^{-rt}} \Rightarrow P(t) = \frac{M}{Ce^{-rt}+1}$$

Initial population: $P(0) = \frac{M}{C+1}$

Long-term (steady-state) population: $\lim_{t\to\infty} P(t) = M$

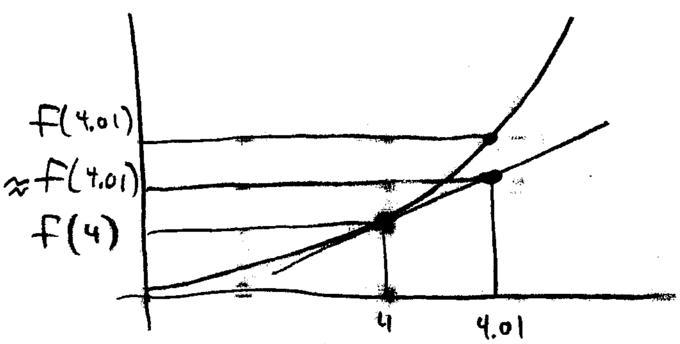
<u>Numerical analysis</u>: The study of approximation & error analysis.

Most ODEs that we actually encounter can't be solved explicitly. But we can solve them numerically.

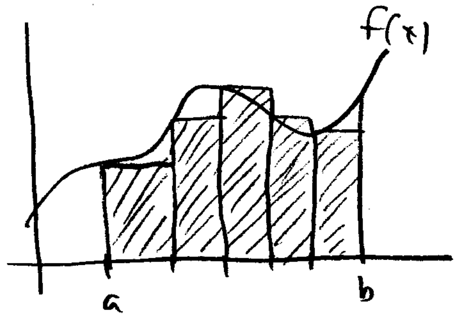
<u>Questions</u>:
- How? (what methods)
- How accurate or reliable is some method?

Recall single-variable calculus:

"approximate the function $f(x)$ at 4.01 using tangent line."



"What is the area under the curve of $f(x)$ b/w a & b?"



(4) **Question:** Consider the ODE $y' = y - t$.

Suppose we couldn't solve it, but we know that $y(1) = 1$.

Can we approximate $y(1.5)$?

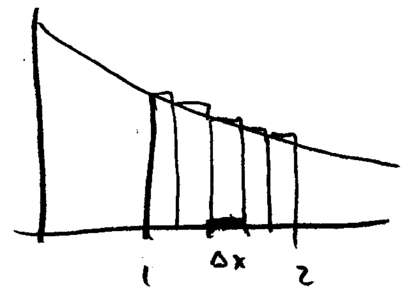
Analogy #1: I give you 2 hrs to come up with the answer to $\int_1^2 e^{-x^2}\,dx$, to 3 decimal places.

Could you do it? And how?

Obviously, you'd use Riemann sums.

But what would you use for Δx?

Would 0.1 work? 0.01? 0.001?



Analogy #2: Find e^3 to 10 decimal places.

How would you do this?

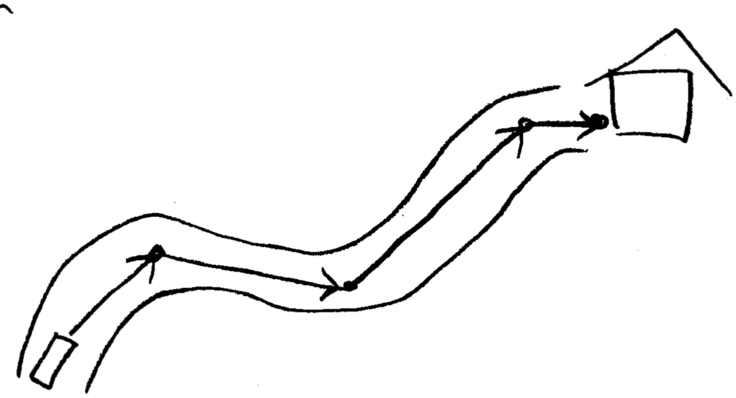
Use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

But how many terms would you need to keep?

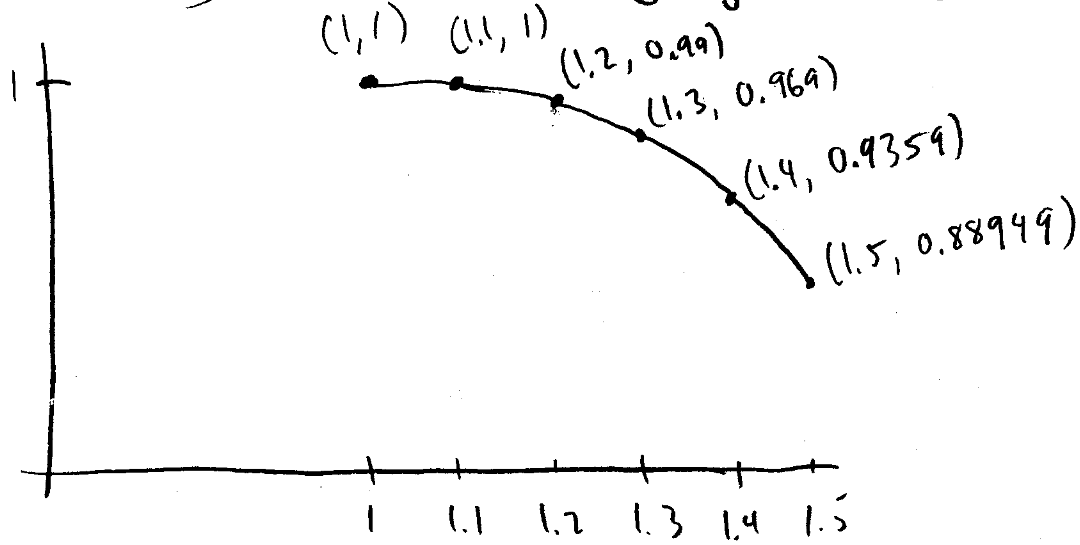
These are questions in Numerical analysis.

Euler's method, pictorially.

Suppose we want to steer a robot down a winding path.



Revisiting our example: $y' = y - t$, $y(1) = 1$.



Start at $(t_0, y_0) = (1, 1)$. Compute $y' = y - t = 1 - 1 = 0$.

Sketch the segment w/ slope 0.

Now, we're at $(t_1, y_1) = (1.1, 1 + 0h) = (1.1, 1)$.

Compute $y' = y - t = 1 - 1.1 = -0.1$

Sketch the segment w/ slope $m = -0.1$

Now, we're at $(t_2, y_2) = (1.2, 1 + mh) = (1.2, 1 + (-\frac{1}{10})(\frac{1}{10}))$
$= (1.2, 0.99)$

Recompute $y' = y - t = 0.99 - 1.2 = -0.21$.

Now, we're at $(t_3, y_3) = (t_3 + h, y_2 + mh) = (1.3, 0.99 + (-0.21)(\frac{1}{10}))$
$= (1.3, 0.969)$

Recompute $y' = y - t = 0.969 - 1.3 = -0.331$.

Now, we're at $(t_4, y_4) = (t_3 + h, y_3 + mh) = (1.4, 0.969 + (-0.331)(\frac{1}{10}))$
$= (1.4, 0.9359)$

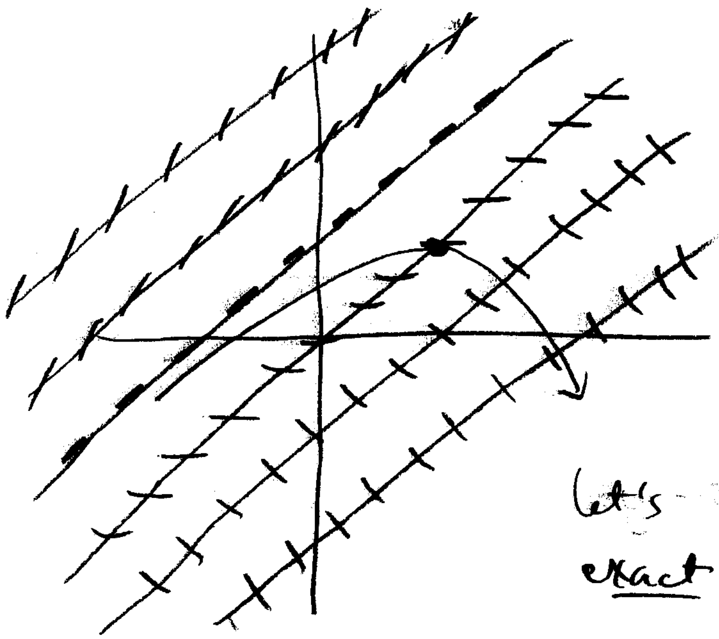
Recompute $y' = y - t = 0.9359 - 1.4 = -0.4641$.

$$(x_5, y_5) = (x_4 + h,\ y_4 + mh) = \left(1.5,\ 0.9359 + (-0.4641)\left(\tfrac{1}{10}\right)\right)$$
$$= (1.5,\ 0.88949)$$

We've determined that $\boxed{y(1.5) \approx 0.88949}$

<u>Question:</u> How good of an approximation is this?

let's sketch some isoclines!



$y' = y - t$

$m = y - t$

$\boxed{y = t + m}$

$m = 0 \Rightarrow y = t$

$m = 1 \Rightarrow y = t + 1$

$m = 2 \Rightarrow y = t + 2$

let's solve this, and find the <u>exact</u> solution:

$y' = y - t \Rightarrow y' - y = -t$ int. factor e^{-t}

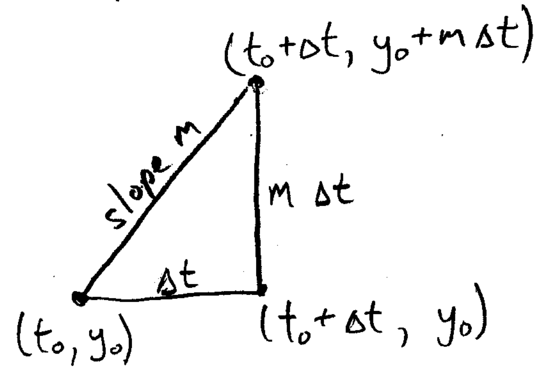
$$(ye^{-t})' = -te^{-t}$$
$$ye^{-t} = e^{-t}(t+1) + C \Rightarrow \boxed{y(t) = Ce^{t} + t + 1}$$

Suppose $y(1) = 1$: $y(1) = Ce + 2 = 1 \Rightarrow C = -\frac{1}{e}$

$$y(t) = -e^{-t}e^{t} + t + 1$$
$$\boxed{y(t) = -e^{t-1} + t + 1}$$
$$\boxed{y(1.5) \approx 0.85128}$$

Euler's method (summary)

Given: $y' = f(t, y)$, $y(t_0) = y_0$ step size h



$$(t_1, y_1) = (t_0+h,\ y_0 + f(t_0, y_0)\cdot h)$$

$$(t_2, y_2) = (t_1+h,\ y_1 + f(t_1, y_1)\cdot h$$

$$\vdots$$

$$(t_{k+1}, y_{k+1}) = (t_k+h,\ y_k + f(t_k, y_k)\cdot h)$$

Chapter 4: 2^{nd} order ODEs

We will consider equations of the form $y'' = f(t, y, y')$

A solution is a function $y(t)$ s.t $y''(t) = f(t, y(t), y'(t))$

Example: $F = ma$ (Newton's second law).

Force (could be gravitational, mechanical, etc, is a function of time, displacement $x(t)$, & velocity $x'(t)$).

$F(t, x, x') = m\,x''(t)$.

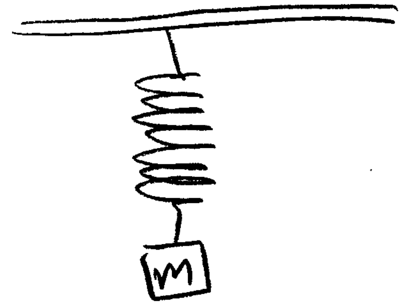
e.g 1: Gravity ("constant" force): $mx''(t) = -mg$.

e.g. 2: Spring  (at rest)

Hooke's law: restoring force $R(x) = -kx$.

Think: "force is proportional to how much we stretch/compress.

$F = \boxed{mx'' = -kx}$



Now, suppose the weight is hanging:

Forces add, so $F = R(x) + mg$

$$\boxed{mx'' = -kx + mg}$$

Now, suppose there's a damping force (springs never "bounce forever")

This is like air resistance:
- proportional to velocity
- acts against the direction of motion.

Thus $D(x') = -\mu x'$

Again, forces add, so $F = D(x') + R(x) + mg$

$$\boxed{mx'' = -\mu x' - kx + mg}$$

Linear 2nd order ODE: $y'' + p(t)\,y' + q(t)\,y = g(t)$

Homogeneous (linear) 2nd order ODE: $y'' + p(t)\,y' + q(t)\,y = 0$.

Analogy: "constant" term is zero
"goes through the origin"
$(t, y(t)) = (0,0)$ is a solution

} whichever one of these you like.

Solutions to 1st order linear ODEs

$$y' = a(t)y + f(t) \quad \text{has a } \underline{\text{one-parameter family}} \text{ of sol'ns,}$$

e.g.: $Ce^t + t^2$.

If we specify $y(t_0) = y_0$, we get a unique <u>particular</u> solution

Solutions to 2nd order linear ODEs:

$$y'' = a(t)y' + b(t)y + f(t) \quad \text{has a } \underline{\text{two-parameter family}} \text{ of sol'ns.}$$

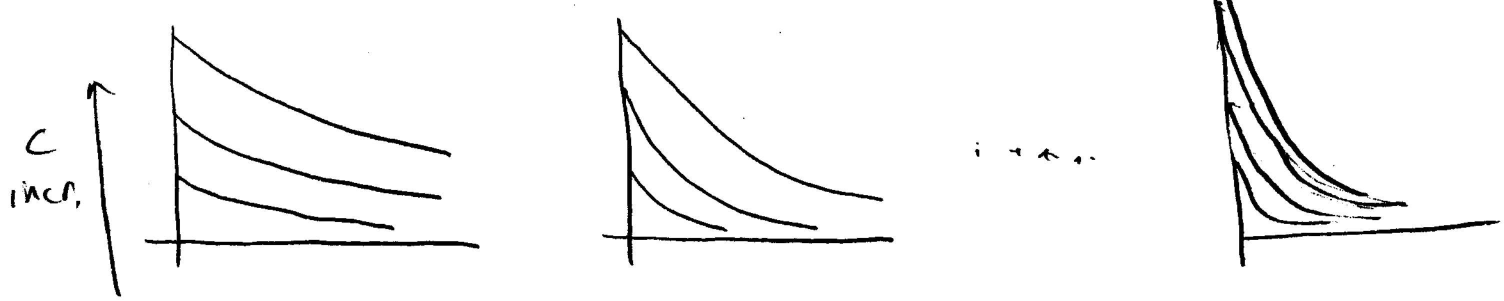
e.g.: $C_1 e^t + C_2 e^{-2t} + t$

If we specify $\begin{cases} y(t_0) = y_0 \\ y'(t_1) = y_1 \end{cases}$ we get a unique <u>particular</u> sol'n

<u>Note</u>: we can also specify $\begin{cases} y(t_0) = y_0 \\ y(t_1) = y_1 \end{cases}$ or $\begin{cases} y'(t_0) = y_0 \\ y'(t_1) = y_1 \end{cases}$

<u>Big idea</u>: Need one initial condition for each constant.

<u>Analogy</u>: $y(t) = Ce^{-kt}$



⟶ k increasing ⟶

(i) Pick $y(0) = 1$. Determines which "curve" we're on (starting point)

(ii) Pick $y'(1) = -3$. Determines which "graph"/axis we're on.

<u>Note</u>: Instead of (ii), we could have said $y(1) = 1/2$.

Another analogy: let v_1, v_2, w be "distinct" vectors

$c v_1 + w$ is a line (through origin if $w=0$)

$c v_1 + c_2 v_2 + w$ is a plane (through $\vec{0}$ if $w=0$).

- As we saw, 1st order linear homog. ODE has sol'n: $C y_h(t)$
  "line through origin"
- 1st order linear ODE has sol'n $C y_h(t) + y_p(t)$.
  "line not through origin"
- 2nd order linear homog. ODE has sol'n $C_1 y_1(t) + C_2 y_2(t)$.
  "plane through origin"
- 2nd order linear ODE has sol'n $C_1 y_1(t) + C_2 y_2(t) + y_p(t)$.
  "plane not through origin"

THIS IS NOT A COINCIDENCE!!!

Application: $\boxed{y'' = ky}$

Simple harmonic motion $\boxed{my'' = -ky}$

How can we solve this?

Write as $\underbrace{y'' = -\frac{k}{m} y}_{\text{what could } y \text{ be?}}$. Compare to $\underbrace{y' = ky}_{\text{exponential function}}$

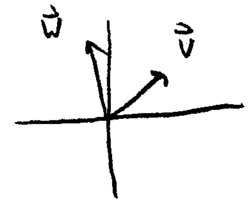
Guess #1: $\cos\left(\sqrt{\frac{k}{m}}\, x\right)$ Guess #2: $\sin\left(\sqrt{\frac{k}{m}}\, x\right)$.

General solution: $A \cos\left(\sqrt{\frac{k}{m}}\, x\right) + B \sin\left(\sqrt{\frac{k}{m}}\, x\right)$

Are these all the solutions? Why/why not?

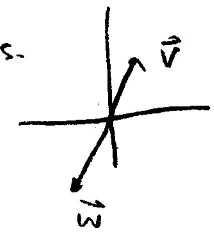
<u>Think:</u> What if we get a general solution

$y(t) = C_1 t^2 + C_2 3t^2$



Are there "really" 2 distinct solutions?

(i.e., a <u>fundamental pair</u> of solutions). vs.



No! But how do we verify this?

<u>Theorem:</u> Let $y(t) = C_1 f(t) + C_2 g(t)$. Then f & g are distinct solutions iff

$$\underbrace{\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}}_{\text{"determinant"}} \neq 0$$

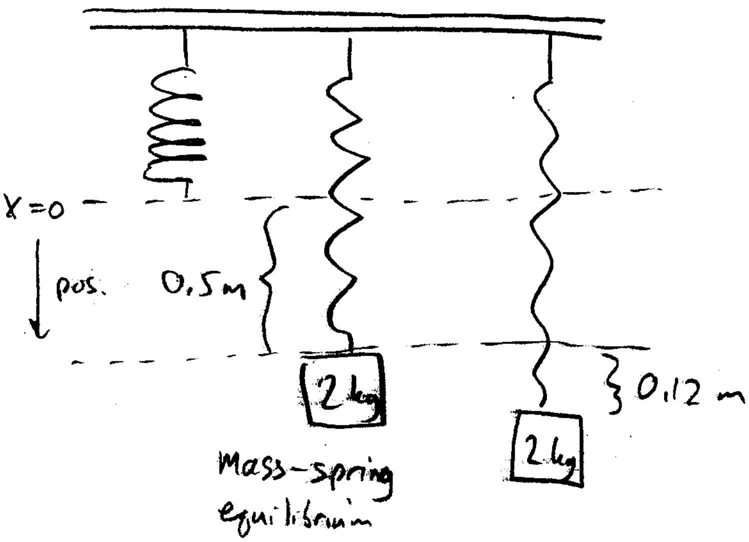
$= f(t)\, g'(t) - f'(t)\, g(t)$

<u>ex:</u>

$$\begin{vmatrix} t^2 & 3t^2 \\ 2t & 6t \end{vmatrix} = 6t^3 - 6t^3 = 0$$

This determinant is called the <u>Wronskian</u>.

Example: 2 kg mass is suspended from a spring. Displacement of the spring-mass equilibrium is 50 cm. If the mass is displaced 12 cm downward from equilibrium, set up the initial value problem that models this (assume no damping).



- 1st determine the spring constant: $kx_0 = mg$

$$\Rightarrow k = \frac{mg}{x_0} = \frac{2 \cdot 9.8}{0.5} = 39.2 \text{ N/m}$$

- 2nd: $F = mx'' = \Sigma$ forces

$$\underbrace{mx''}_{\text{total force}} = \underbrace{-\mu x'}_{\substack{\text{damping}\\(=0)}} \underbrace{-kx}_{\text{spring}} + \underbrace{mg}_{\text{grav.}} + \underbrace{F(t)}_{\substack{\text{driving}\\(=0)}}$$

$$mx'' = -kx + mg$$

$$2x'' = -kx + kx_0 = -k(x - x_0) = -39.2(x - 0.5)$$

$$\boxed{2x'' + 39.2(x-0.5) = 0, \quad x(0) = 0.62, \quad x'(0) = 0}$$

Let's make a change of variables. (b/c we don't know how to solve this!)

Let $y = x - 0.5$, $y'' = x''$

$$\boxed{\text{Now, we get } 2y'' + 39.2y, \quad y(0) = 0.12, \quad y'(0) = 0}$$

Let's solve this: $y' = -19.6y$ Let $\omega = \sqrt{19.6}$

$$y(t) = A\cos\omega t + B\sin\omega t$$

$$y'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$y'(0) = 0 + B\omega = 0 \Rightarrow B = 0.$$

$$y(x) = A\cos\omega t$$

$$y(0) = A = 0.12 \Rightarrow y(t) = 0.12\cos\omega t.$$

Recall: $y(t) = X(t) - 0.5 \Rightarrow \boxed{X(t) = 0.12\cos(\omega t) + 0.5}$